Linear Programming: Gaussian Elimination

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Translate between systems of equations and augmented matrices.
- Row reduce a matrix.
- Write an algorithm to solve linear systems.

Last Time

Solving systems of linear equations by substitution.

Notation

To simplify notation, instead of writing full equations like

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We just store coefficients of equations in an (augmented) matrix, like so:

$$\begin{matrix} x & y & = 1 \end{matrix}$$

$$\left[\begin{array}{cc|r} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}\right]$$

Substitution

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Subtract twice first row from second.

Basic Row Operations

There are three basic ways to manipulate our matrix. These are called Basic row operations. Each of them gives us an equivalent system of equations.

Adding

Add/subtract a multiple of one row to another.

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right]$$

Scaling

Multiply/divide a row by a non-zero constant.

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Multiply/divide a row by a non-zero constant. Dividing the second row by 2:

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

Swapping

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Sometimes you want to change the ordering of rows. For example, swapping the first and second rows we get

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 5 \end{array}\right]$$

Row Reduction

Row reduction uses row operations to put a matrix into a simple standard form. The idea is to simulate the substitution method.

Example

Consider the system given by the matrix:

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Use first for to solve for first variable.

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Divide first row by 2.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Substitute into other equations.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Add first row to second.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Subtract twice first row from third.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

Example

Need to solve for next variable.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

Example

Cannot use second row.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

Example

Swap second and third rows.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Solve for second variable.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Divide second row by -2.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Substitute into other equations.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Subtract twice second row from first.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Can't solve for third variable.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Solve for fourth instead.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Divide last row by -2.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Substitute into other equations.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Subtract twice third row from first.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Add third row to second.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Done.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Answer

Our matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

corresponds to equations:

$$\begin{aligned} x + z &= -1 \\ y - z &= 1 \\ w &= 0. \end{aligned}$$

Solution

So for any value of z, we have solution:

$$\begin{aligned} x &= -1 - z \\ y &= 1 + z \\ w &= 0. \end{aligned}$$

RowReduce(A)

Leftmost non-zero

Swap row to top

Make entry pivot

Rescale to make pivot 1

Subtract row from others to make other entries in column 0

Repeat

RowReduce(A)

Leftmost non-zero in non-pivot row
Swap row to top of non-pivot rows
Make entry pivot
Rescale to make pivot 1
Subtract row from others to make
other entries in column 0
Repeat until no more non-zero
entries outside of pivot rows

Reading off Answer

- Each row has one pivot and a few other non-pivot entries.
- Gives equation writing pivot variable in terms of non-pivot variables.
- If pivot in units column, have equation $0 = 1$, so no solutions.
- Otherwise, set non-pivot variables to anything, gives answer.

Runtime

- m equations in n variables.
- $\min(n, m)$ pivots.
- For each pivot, need to subtract multiple of row from each other row $O(nm)$ time.
- Total runtime: $O(nm \min(n, m))$.