Linear Programming: Solving Linear Systems

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Advanced Algorithms and Complexity
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Learning Objectives

- Solve a system of linear equations.
- Say something about what the set of solution to a system of linear equations looks like.

Last Time

Linear programming: Dealing with systems of linear inequalities.

Linear Algebra

Today, we will deal with the simpler case, of systems of linear equalities.

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For example:

$$
\begin{aligned}
x + y &= 5 \\
2x + 4y &= 12.
\end{aligned}
$$

Method of Substitution

- Use first equation to solve for one variable in terms of the others.
- Substitute into other equations.
- Solve recursively.
- Substitute back in to first equation to get initial variable.

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So $y = 1, x = 5 - 1 = 4.$

Problem

What is the value of x in the solution to the following linear system?

$$\begin{aligned} x + 2y &= 6 \\ 3x - y &= -3. \end{aligned}$$

Solution

From the first equation, we get

$$x = 6 - 2y.$$

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Solving gives, $y = 3$, so $x = 6 - 2 \cdot 3 = 0$.

Another Example

Consider the following system of equations:

$$\begin{aligned} x + y + z &= 5 \\ 2x + y - z &= 1. \end{aligned}$$

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Solve by substitution.

Solution

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Substitute into second.

$$2(5 - y - z) + y - z = 1,$$

or

$$y = 9 + 3z.$$

Cannot Solve for z!

No equations left.

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No equations left.However, for any z have solution

$$\begin{aligned} y &= 9 + 3z \\ x &= 5 - y - z = -4 - 4z. \end{aligned}$$

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Have entire family of solutions. z is a free variable.

Degrees of Freedom

- Your solution set will be a subspace.
- Dimension = number of free variables.
- Each equation gives one variable in terms of others.
- Generally, dimension equals

$$\text{num. variables} - \text{num. equations}.$$

Summary

- Can solve systems using method of substitution.
- Each equation reduces degrees of freedom by one.

Next Time

Systematize this to simplify notation and make into an algorithm.