Linear Programming: Linear Programming

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Understand the formal definition of a linear programming problem.
- Provide some examples of linear programming problems

Last Time

Factory. Set M, W to maximize $200M + 100W$ subject to

- $W \geq 0$.
- $100 \geq M \geq 0$.
- $W \geq 2M$.
- $100,000 \geq 200(W - 2M) + 600M$.

Linear Programming

Linear programming asks for real numbers $x_1, x_2, \ldots, x_n$ satisfying linear inequalities:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\geq b_1 \\ &\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\geq b_m \end{aligned}$$

So that a linear objective

$$v_1x_1 + v_2x_2 + \ldots + v_nx_n$$

is as large (or small) as possible.

Notation

Linear Programming

Input: An $m \times n$ matrix A and vectors $b \in \mathbb{R}^m, v \in \mathbb{R}^n$

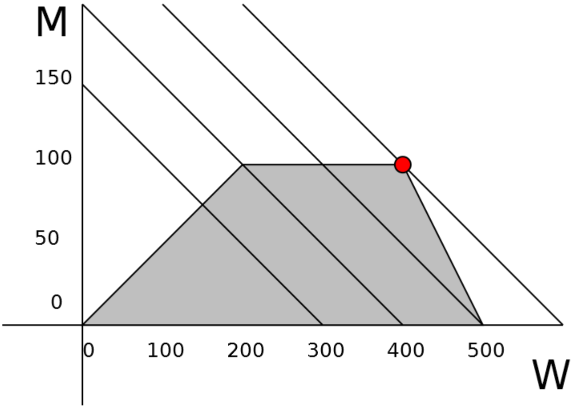
Output: A vector $x \in \mathbb{R}^n$ so that $Ax \geq b$ and $v \cdot x$ is as large (or small) as possible.

Examples

Linear programming is useful because an extraordinary number of problems can be put into this framework.

Factory Example

The factory example we just worked.



The Diet Problem

Studied by George Stigler in the 1930s and 1940s.

How cheaply can you purchase food for a healthy diet?

Variables

You have a number of types of food (bread, milk, apples, etc.).
For each you have a variable giving the number of servings per day.

$$x_{bread}, x_{milk}, x_{apples}, \ldots$$

Constraints

Non-negative number of servings:

$$x_f \geq 0.$$

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Sufficient calories/day:

$$\begin{aligned}&(\text{Cal/serving bread})x_{bread}\\ +&(\text{Cal/serving milk})x_{milk} + \ldots \geq 2000.\end{aligned}$$

Constraints

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Similar constraints for other nutritional needs (vitamin C, protein, etc.)

Optimization

Minimize cost.

$$\begin{aligned}&(\text{cost of serving bread})x_{bread}\\&+(\text{cost of serving milk})x_{milk} + \ldots\end{aligned}$$

Optimization

Minimize cost.

$$\text{(cost of serving bread)}x_{bread}$$
$$+\text{(cost of serving milk)}x_{milk} + \dots$$

Warning: actually doing this can get you some pretty weird diets.

Network Flow

Network flow problems are actually just a special case of linear programming problems!

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Variables: f_e for each edge e.

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$$0 \leq f_e \leq C_e.$$

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Objective:

$$\sum_{e \text{ out of } s} f_e - \sum_{e \text{ into } s} f_e$$

Strange Cases

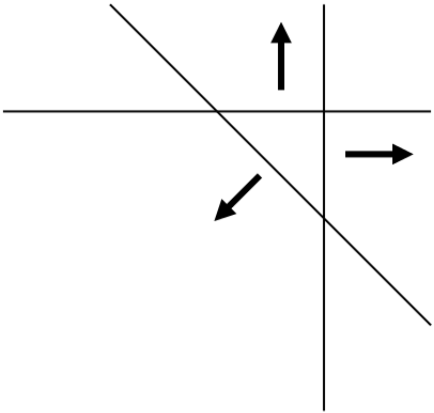
There are a couple of edge cases to keep in mind here.

- No Solution
- No Optimum

No Solution

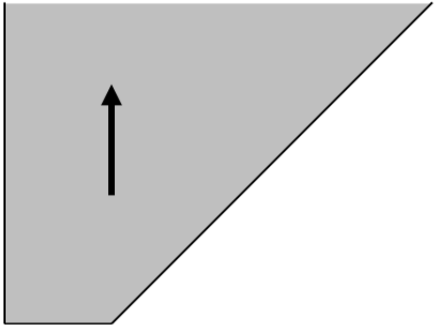
Consider the system:

$x \geq 1, \quad y \geq 1, \quad x + y \leq 1.$



No Optimum

Consider trying to maximize x subject to $x \geq 0$, $y \geq 0$, and $x - y \geq 1$.



Problem

Of these three systems, one has no solution, one has no maximum x value, and one has a maximum. Which is which?

(A) $x + y \geq 1$, $x + y \leq 0$.

(B) $x + y \leq 2$, $x - y \leq 1$.

(C) $x + y \geq 0$, $x - y \leq 0$.

Solution