Linear Programming: Duality

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Write the dual program of a linear program.
- Understand the duality theorem.

Example

Recall our first example:

Maximize $200M + 100W$ subject to:

- $W \geq 0$.
- $100 \geq M \geq 0$.
- $W \geq 2M$.
- $100,000 \geq 200(W - 2M) + 600M$.

Upper Bound

The best you could do was 60000, but we proved it by combining constraints

$$\begin{array}{rl} 100 \cdot [& 001 \cdot M + 000 \cdot W \ \leq 100] \\ +0.5 \cdot [& 200 \cdot M + 200 \cdot W \ \leq 100,000] \\ \hline & 200 \cdot M + 100 \cdot W \ \leq 60,000. \end{array}$$

General Technique

Try to prove bound by combining the constraints together.

Linear Program

Say you have the linear program where you want to minimize

$$v_1x_1 + v_2x_2 + \ldots + v_nx_n$$

subject to constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &\geq b_1 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &\geq b_m \end{aligned}$$

Combine Constraints

If we have $c_i \geq 0$, we can combine constraints:

$$
\begin{array}{rcl}
c_1 \cdot [a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & \geq & b_1] \\
\cdots & & \\
+c_m \cdot [a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & \geq & b_m] \\
\hline
w_1x_1 + w_2x_2 + \ldots + w_nx_n & \geq & t,
\end{array}
$$

$w_i = \sum c_j a_{ji}$, $t = \sum c_j b_j$.

Bound

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Want to find $c_i \geq 0$ so that $v_i = \sum_{j=1}^{m} c_j a_{ji}$ for all i, and $t = \sum_{j=1}^{m} c_j b_j$ is as large as possible.

Linear Program

Note that this is another linear program.
Find $c \in \mathbb{R}^m$ so that $\sum_{j=1}^m c_j b_j$ is as large as possible, subject to the linear inequalities $c_i \geq 0$, and equalities

$$v_i = \sum_{j=1}^{m} c_j a_{ji}.$$

Dual Program

Definition

Given the linear program (the primal):

Minimize $v \cdot x$

Subject to $Ax \geq b$

The dual linear program is the linear program:

Maximize $y \cdot b$

Subject to $y^T A = v$, and $y \geq 0$.

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The surprising thing is that these two linear programs always have the same solution.

Duality

Theorem

A linear program and its dual always have the same (numerical) answer.

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This means that one can instead solve the dual problem. This is sometimes easier, and often provides insight into the solution.

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By adding multiples of the conservation of flow equation, this is

$$\sum_v c_v \left(\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right)$$

where $c_s = 1, c_t = 0$.

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We can bound this using capacity constraints as
$$\sum_{e=(v,w)} C_e \max(c_v - c_w, 0).$$

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The dual program just finds the minimum cut!

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Recall the diet problem:

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What is the dual program?

Example: Diet Problem

For each nutrient N, use a multiple C_N of the equation for that nutrient.
Can then add multiples of the constraint that you get a non-negative amount of each food.

Example: Diet Problem

Think of C_N as a cost of nutrient N. We pick values so that for each food item, f, we have

$$\text{Cost}(f) \geq \sum_N C_N \cdot (\text{Amount of nutrient } N \text{ in } f).$$

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Costs C_N to get a unit of nutrient N. This means total cost of a balanced diet is at least

$$\sum_N C_N \cdot (\text{Required amount of nutrient } N).$$

Observation

Note that if you want to actually obtain this lower bound, you cannot buy overpriced foods. Can only afford to buy foods with

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This is an example of a general phenomena called complementary slackness.

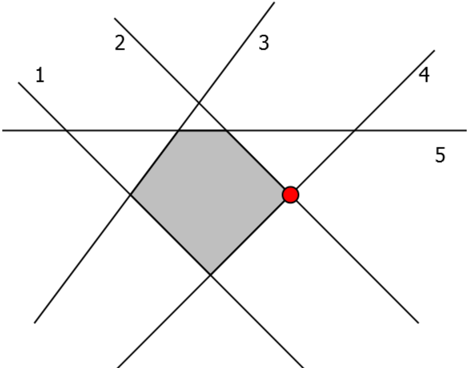
Complementary Slackness

Theorem

Consider a primal LP:
Minimize $v \cdot x$ subject to $Ax \geq b$,
and its dual LP:
Maximize $y \cdot b$ subject to $y^T A = v$, $y \geq 0$.
Then in the solutions, $y_i > 0$ only if the i^{th} equation in x is tight.

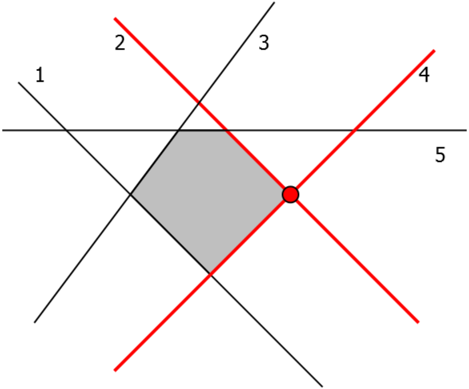
Problem

Assuming that the highlighted point is the optimum to the linear program below, which equations might have non-zero coefficients in the solution to the dual program?



Solution

Only 2 and 4.



Summary

- Every LP has dual LP.
- Solutions to dual bound solutions to primal.
- LP and dual have same answer!
- Complementary slackness.