NP-complete Problems: Reductions

Alexander S. Kulikov

Steklov Institute of Mathematics at St. Petersburg
Russian Academy of Sciences

Advanced Algorithms and Complexity
Data Structures and Algorithms

Outline

1. Reductions
2. Showing **NP**-completeness
3. Independent Set $\rightarrow$ Vertex Cover
4. 3-SAT $\rightarrow$ Independent Set
5. SAT $\rightarrow$ 3-SAT
6. All of **NP** $\rightarrow$ SAT
7. Using SAT-solvers

Informally

We say that a search problem A is reduced to a search problem B and write $A \to B$, if a polynomial time algorithm for B can be used (as a black box) to solve A in polynomial time.

Reduction: $A \to B$

instance I of A

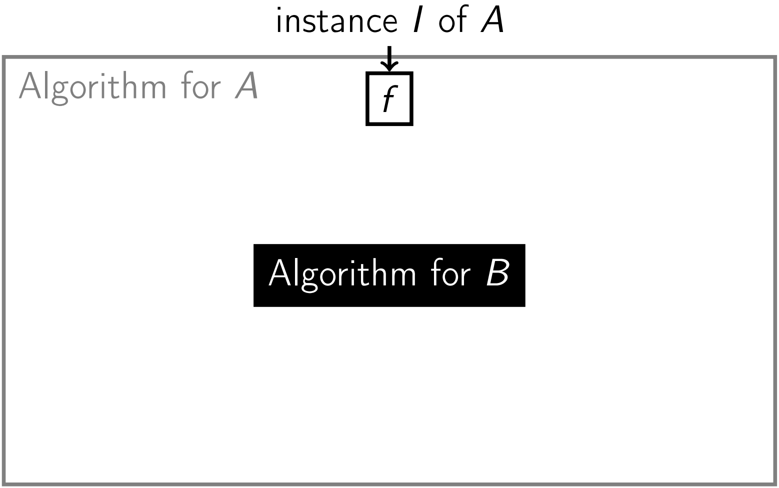
Reduction: $A \to B$

instance I of A

Algorithm for A

Algorithm for B

Reduction: $A \to B$



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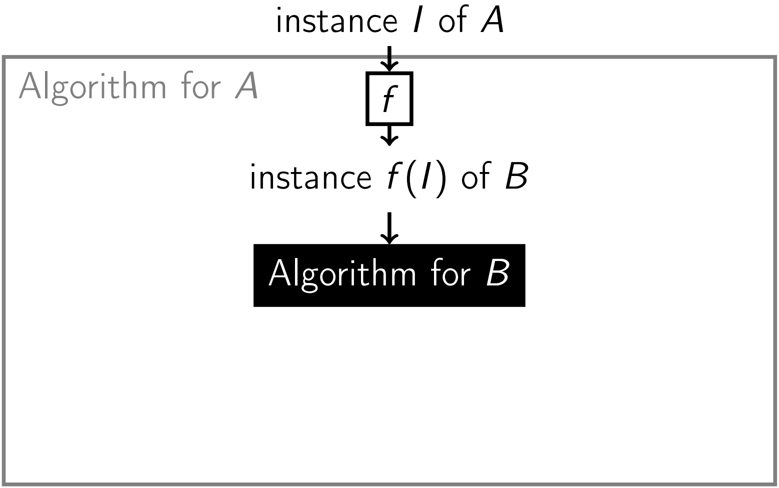

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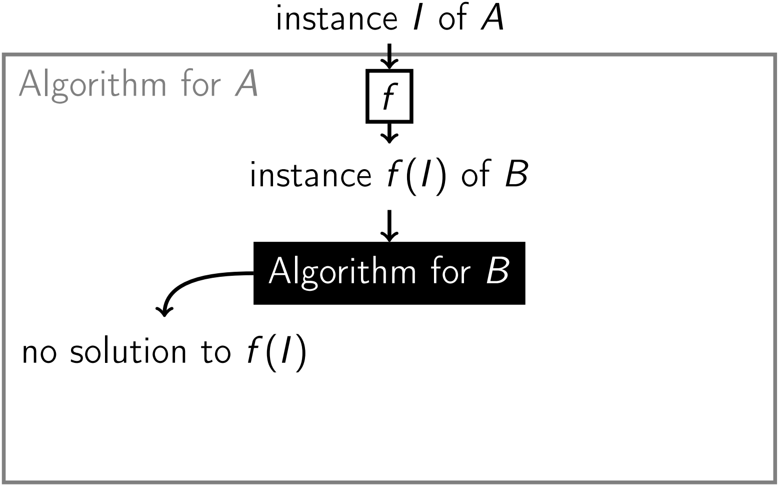

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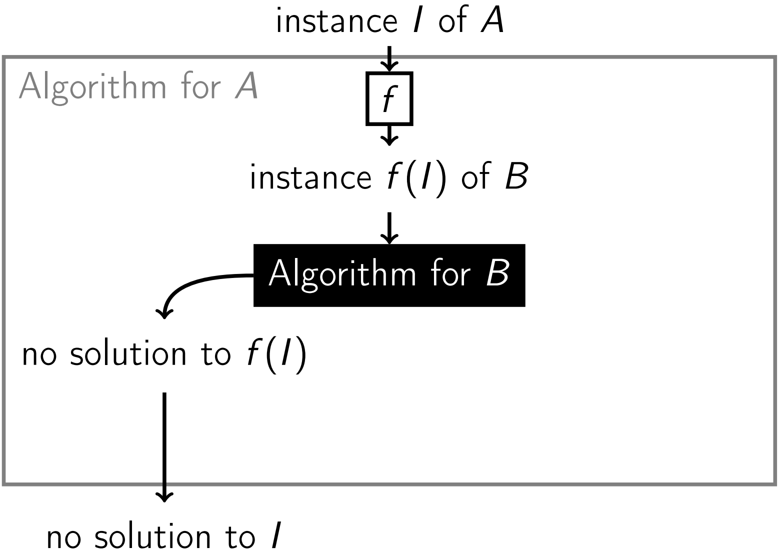

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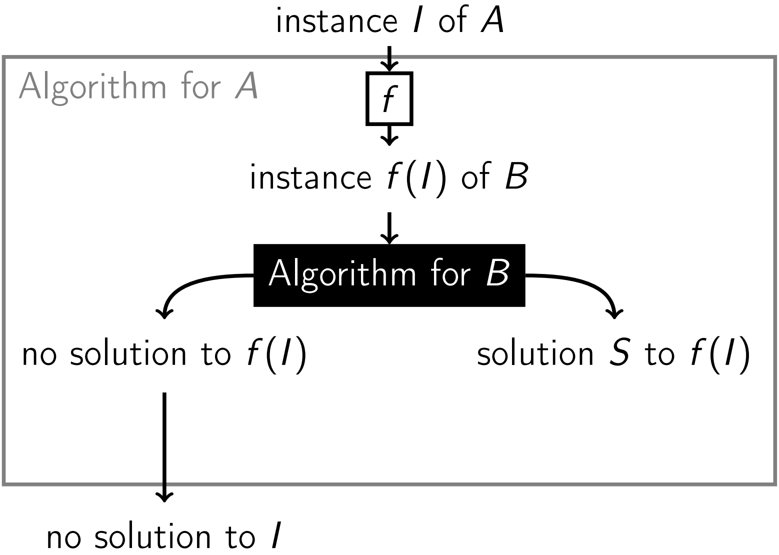

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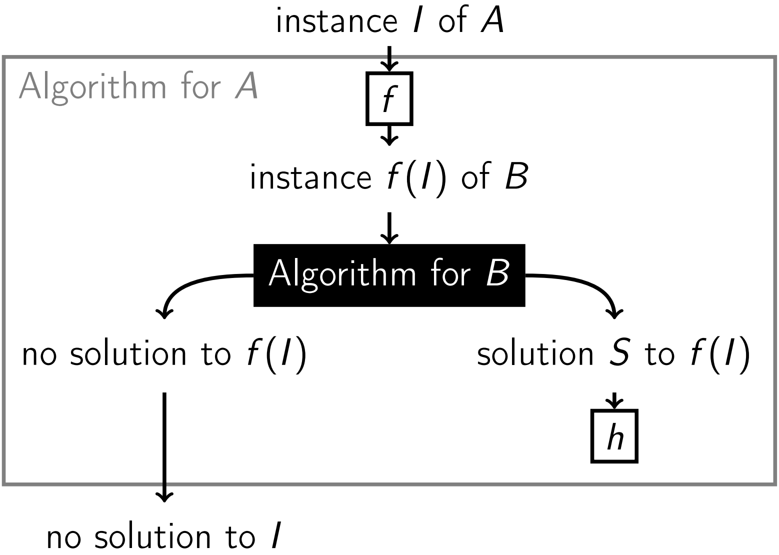

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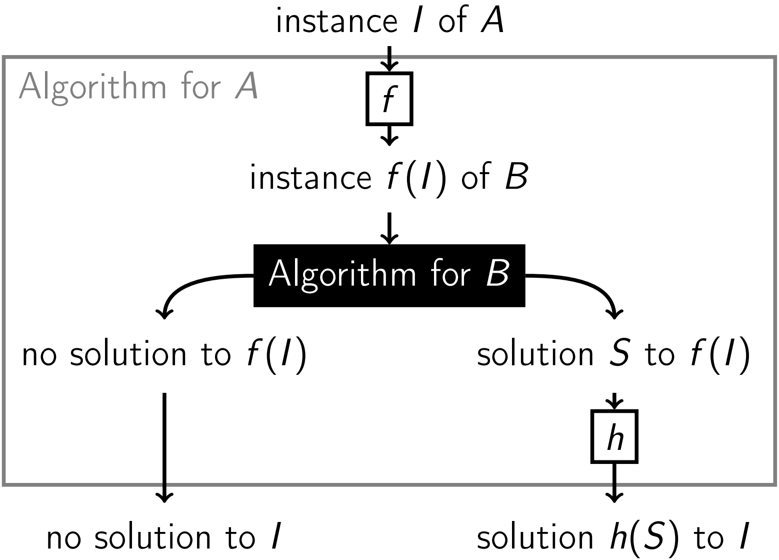
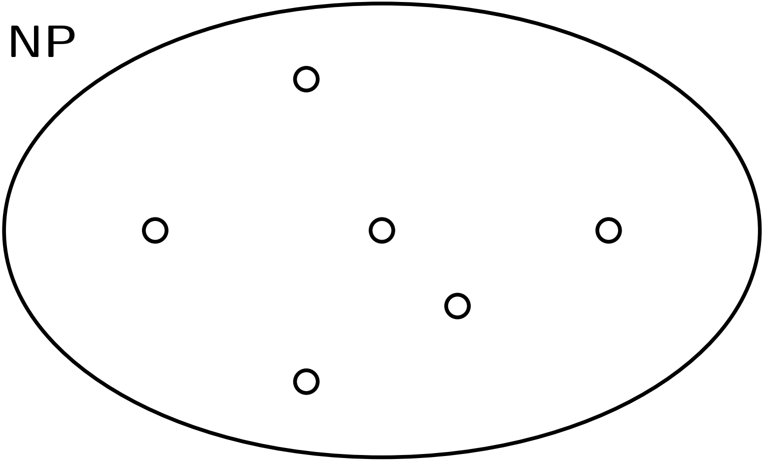


Formally

Definition

We say that a search problem A is reduced to a search problem B and write $A \to B$, if there exists a polynomial time algorithm f that converts any instance I of A into an instance $f(I)$ of B, together with a polynomial time algorithm h that converts any solution S to $f(I)$ back to a solution $h(S)$ to A. If there is no solution to $f(I)$, then there is no solution to I.

Graph of Search Problems



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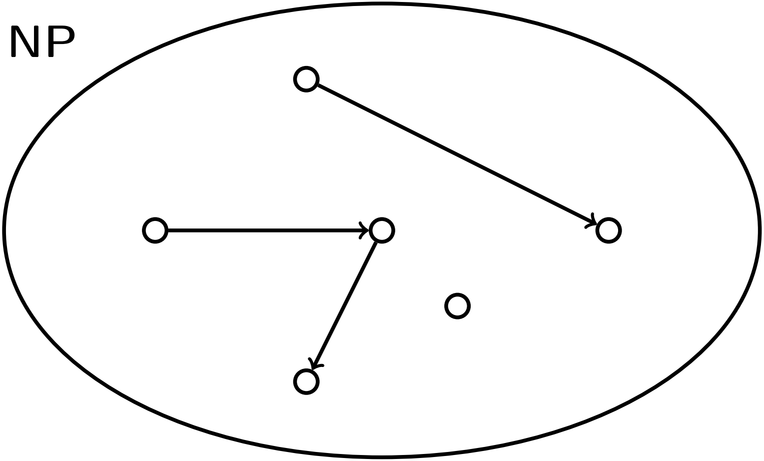

NP-complete Problems

Definition

A search problem is called **NP-complete** if all other search problems reduce to it.

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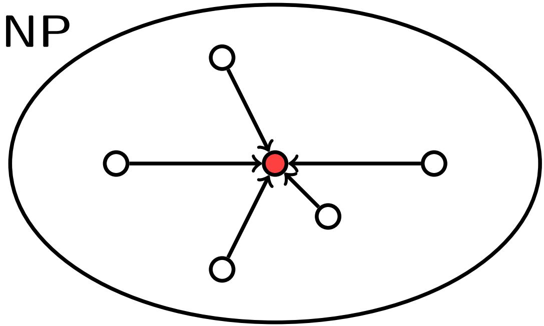

Do they exist?

It's not at all immediate that **NP**-complete problems even exist. We'll see later that all hard problems that we've seen in the previous part are in fact **NP**-complete!

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Two ways of using $A \to B$:

1. if B is easy (can be solved in polynomial time), then so is A
2. if A is hard (cannot be solved in polynomial time), then so is B

Reductions Compose

Lemma

If $A \to B$ and $B \to C$, then $A \to C$.

Proof

- The reductions $A \to B$ and $B \to C$ are given by pairs of polytime algorithms (f_{AB}, h_{AB}) and (f_{BC}, h_{BC}).

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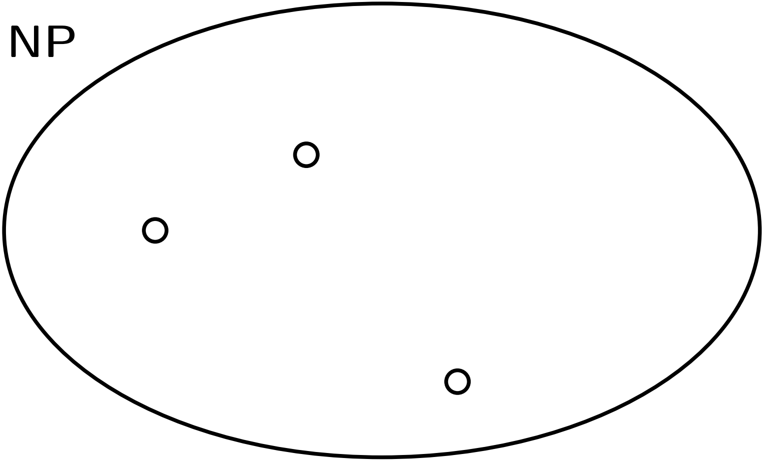
- To transform an instance I_A of A to an instance I_C of C we apply a polytime algorithm $f_{BC} \circ f_{AB}$: $I_C = f_{BC}(f_{AB}(I_A))$.
- To transform a solution S_C to I_C to a solution S_A to I_A we apply a polytime algorithm $h_{AB} \circ h_{BC}$: $S_A = h_{AB}(h_{BC}(S_C))$. □

Pictorially



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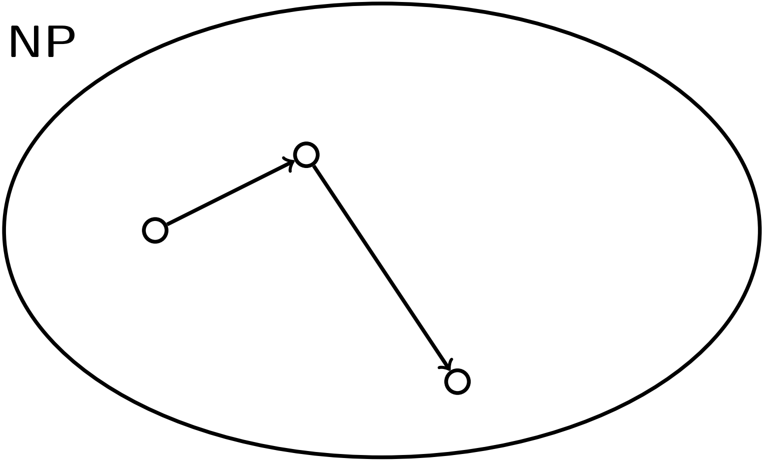

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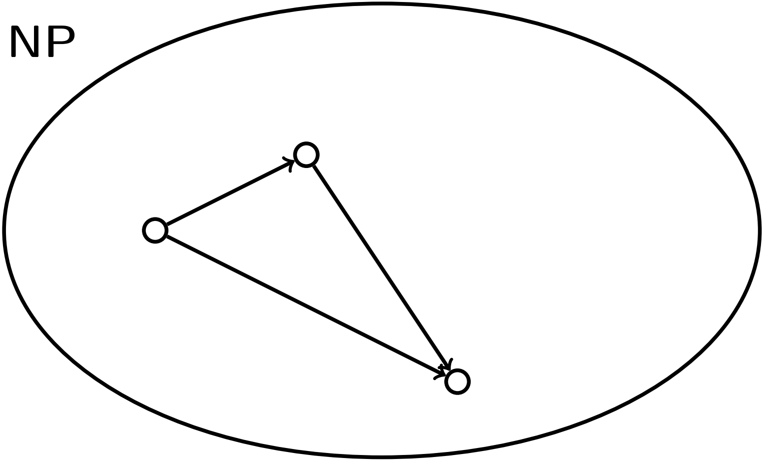

Showing **NP**-completeness

Corollary

If $A \to B$ and A is **NP**-complete, then so is B.

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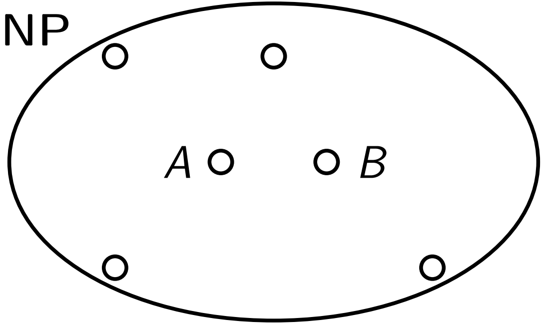

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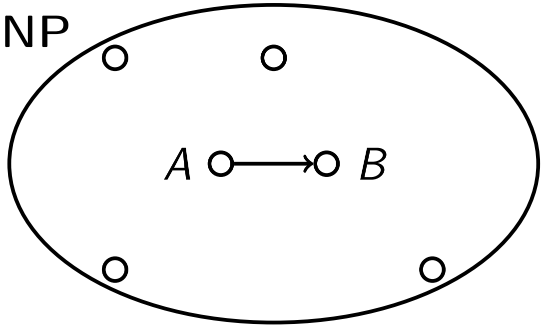
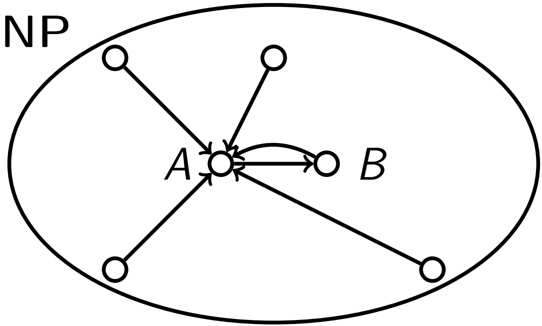


Showing NP-completeness

Corollary

If $A \rightarrow B$ and A is **NP**-complete, then so is B.



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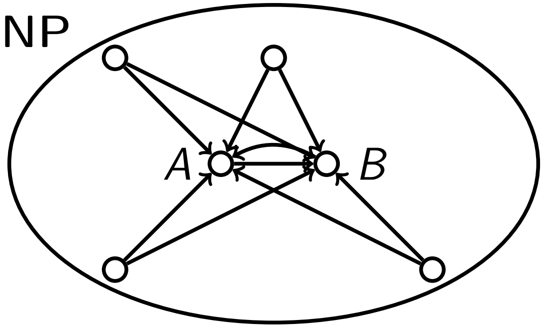
If $A \to B$ and A is **NP**-complete, then so is B.



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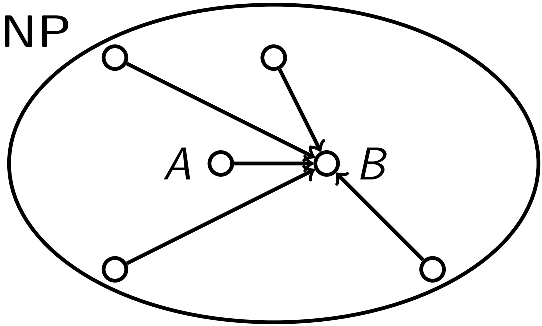
If $A \to B$ and A is **NP**-complete, then so is B.

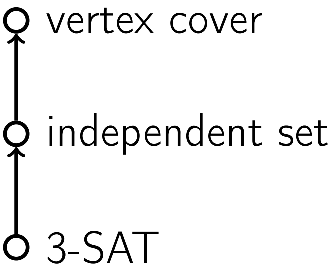


Plan

vertex cover

independent set

Plan



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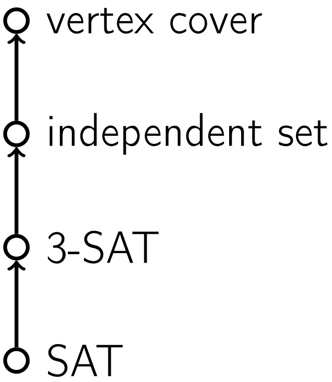

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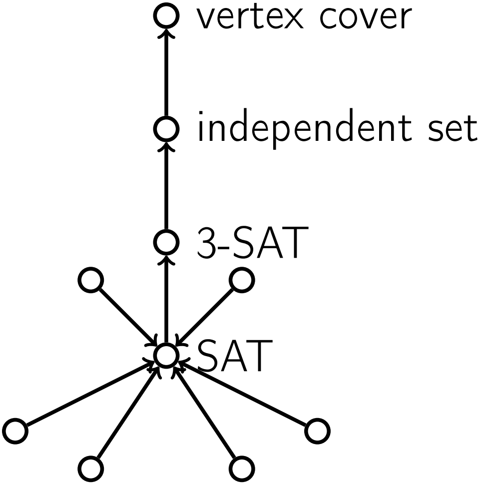

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Independent set

Input: A graph and a budget b.

Output: A subset of at least b vertices such that no two of them are adjacent.

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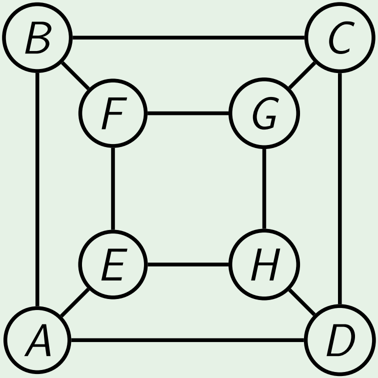
Vertex cover

Input: A graph and a budget b.

Output: A subset of at most b vertices that touches every edge.

Example



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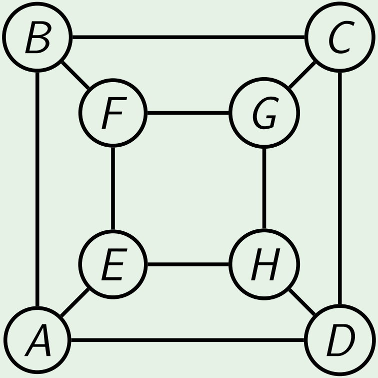
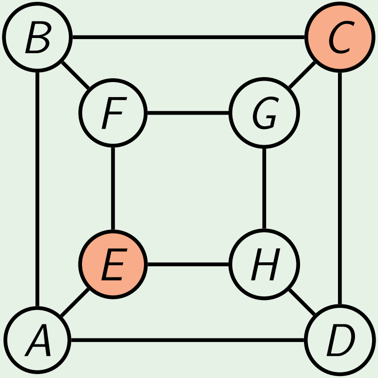


Independent sets:

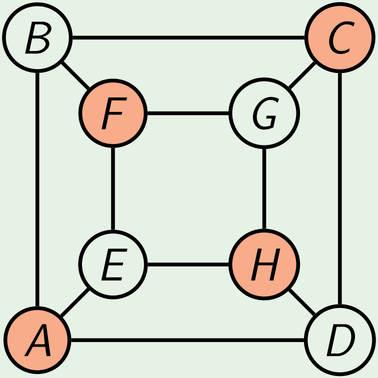
Example



Independent sets:

$\{E, C\}$

Example



Independent sets:

$\{E, C\}$ $\{A, C, F, H\}$

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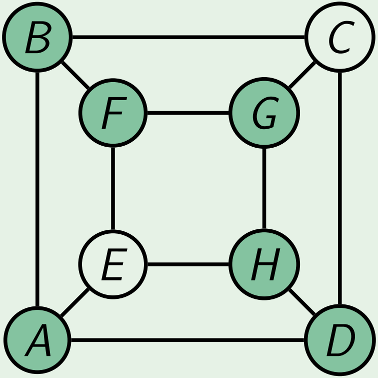

Independent sets:

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Vertex covers:

$\{A, B, D, F, G, H\}$

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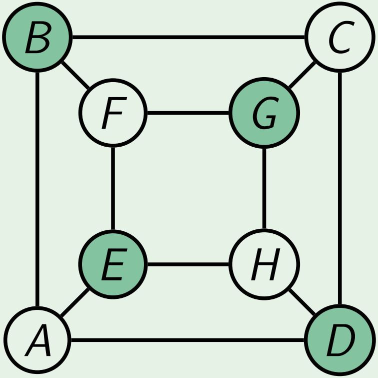

Independent sets:

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$\{B, D, E, G\}$

Lemma

I is an independent set of $G(V, E)$, if and only if $V - I$ is a vertex cover of G.

Proof

- $\Rightarrow$ If I is an independent set, then there is no edge with both endpoints in I. Hence $V - I$ touches every edge.
- $\Leftarrow$ If $V - I$ touches every edge, then each edge has at least one endpoint in $V - I$. Hence I is an independent set. $\square$

Reduction

Independent set $\rightarrow$ vertex cover: to check whether $G(V, E)$ has an independent set of size at least b, check whether G has a vertex cover of size at most $|V| - b$:

- $f(G(V, E), b) = (G(V, E), |V| - b)$
- $h(S) = V - S$

Outline

3-SAT

Input: Formula F in 3-CNF (a collection of clauses each having at most three literals).

Output: An assignment of Boolean values to the variables of F satisfying all clauses, if exists.

Goal

Design a polynomial time algorithm that, given a 3-CNF formula F, outputs a graph G and an integer b, such that:

> *F is satisfiable, if and only if G has an independent set of size at least b.*

We need to find an assignment of Boolean values to variables, such that each clause contains at least one satisfied literal.

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Example

- Setting $x = 1, y = 1, z = 1$ satisfies a formula $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$.

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Example

- Setting $x = 1, y = 1, z = 1$ satisfies a formula $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$.
- Setting $x = 1, y = 0, z = 0$ also satisfies it: $(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})$.

Alternatively, we need to select at least one literal from each clause, such that the set of selected literals is consistent: it does not contain a literal ℓ together with its negation $\overline{\ell}$.

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- Consistent: $\{x, x, \bar{z}\}$, $\{x, x, y\}$, $\{x, \bar{y}, \bar{z}\}$.

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- Consistent: $\{x, x, \overline{z}\}$, $\{x, x, y\}$, $\{x, \overline{y}, \overline{z}\}$.
- Inconsistent: $\{y, \overline{y}, \overline{z}\}$, $\{z, x, \overline{z}\}$.

Using Alternative Statement

$$(x \vee y \vee z)(x \vee \overline{y})(y \vee \overline{z})(z \vee \overline{x})(\overline{x} \vee \overline{y} \vee \overline{z})$$

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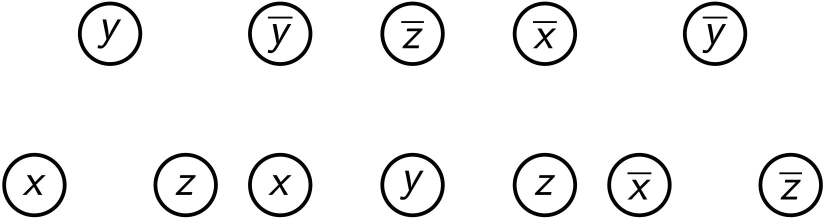

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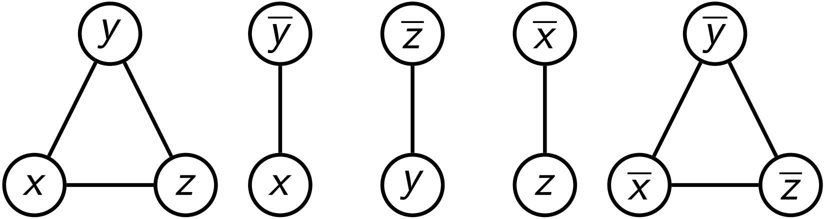

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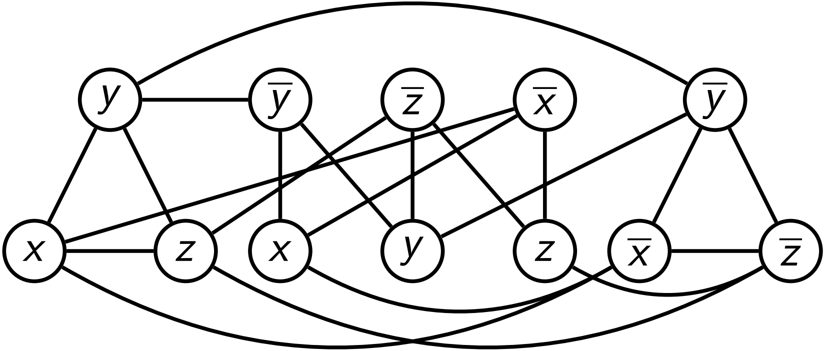

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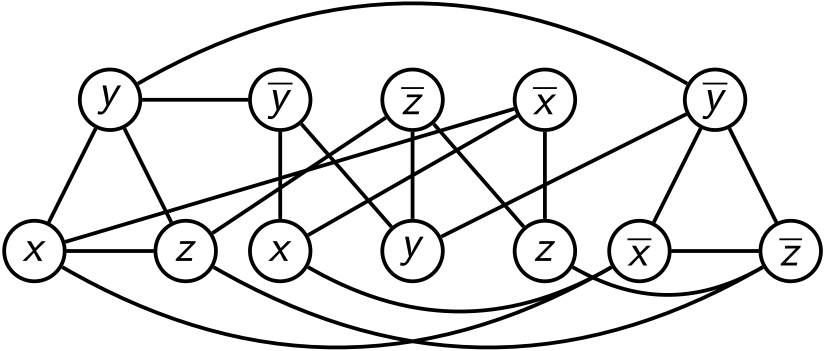

the formula is satisfiable iff the resulting graph has independent set of size 5

Transforming an Instance

- For each clause of the input formula F, introduce three (or two, or one) vertices in G labeled with the literals of this clause. Join every two of them.

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- Join every pair of vertices labeled with complementary literals.
- F is satisfiable if and only if G has independent set of size equal to the number of clauses in F.
- Transformation takes polynomial time.

Transforming a Solution

- Given a solution S for G, just read the labels of the vertices from S to get a satisfying assignment of F (takes polynomial time).

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- Given a solution S for G, just read the labels of the vertices from S to get a satisfying assignment of F (takes polynomial time).
- If there is no solution for G, then F is unsatisfiable: indeed, a satisfying assignment for F would give a required independent set in G.

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Goal

Transform a CNF formula into an equisatisfiable 3-CNF formula. That is, reduce a problem to its special case.

Transforming an Instance

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- We need to get rid of clauses of length more than 3 in an input formula
- Consider such a clause: $C = (\ell_1 \lor \ell_2 \lor A)$, where A is an OR of at least two literals.
- Introduce a fresh variable y and replace C with the following two clauses: $(\ell_1 \lor \ell_2 \lor y)$, $(\overline{y} \lor A)$
- The second clause is shorter than C
- Repeat while there is a long clause

Running time

The running time of the transformation is polynomial: at each iteration we replace a clause with a shorter clause and a 3-clause. Hence the total number of iterations is at most the total number of literals of the initial formula.

Correctness

Lemma

The formulas $F = (\ell_1 \vee \ell_2 \vee A)\ldots$ and $F' = (\ell_1 \vee \ell_2 \vee y)(\overline{y} \vee A)\ldots$ are equisatisfiable.

Proof

$F = (\ell_1 \vee \ell_2 \vee A) \ldots$

$F' = (\ell_1 \vee \ell_2 \vee y)(\overline{y} \vee A) \ldots$

- $\Rightarrow$ If either ℓ_1 or ℓ_2 is satisfied, set $y = 0$. Otherwise A must be satisfied. Then set $y = 1$.
- $\Leftarrow$ If $(\ell_1 \vee \ell_2 \vee y)(\overline{y} \vee A)$ are satisfied, then so is $(\ell_1 \vee \ell_2 \vee A)$. $\square$

Transforming a Solution

Given a satisfying assignment for F', just throw away the values of all new variables (y's) to get a satisfying assignment of the initial formula.

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Show that every search problem reduces to SAT.

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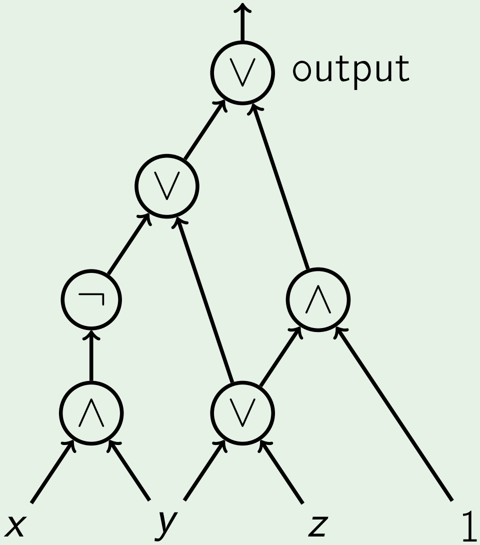
Instead, we show that any problem reduces to Circuit SAT problem, which, in turn, reduces to SAT.

Circuit



Definition

A circuit is a directed acyclic graph of in-degree at most 2. Nodes of in-degree 0 are called inputs and are marked by Boolean variables and constants. Nodes of in-degree 1 and 2 are called gates: gates of in-degree 1 are labeled with NOT, gates of in-degree 2 are labeled with AND or OR. One of the sinks is marked as output.

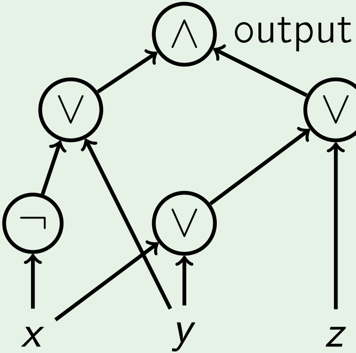
Circuit-SAT

Input: A circuit.

Output: An assignment of Boolean values to the input variables of the circuit that makes the output true.

SAT is a special case of Circuit-SAT as a CNF formula can be represented as a circuit:

Example: $(x \vee y \vee z)(y \vee \overline{x})$



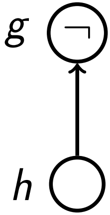
Circuit-SAT → SAT

To reduce Circuit-SAT to SAT, we need to design a polynomial time algorithm that for a given circuit outputs a CNF formula which is satisfiable, if and only if the circuit is satisfiable

Idea

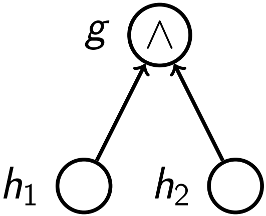
- Introduce a Boolean variable for each gate
- For each gate, write down a few clauses that describe the relationship between this gate and its direct predecessors

NOT Gates



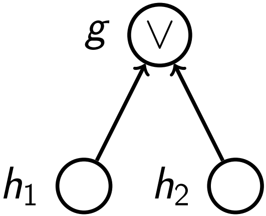
$$(h \vee g)(\overline{h} \vee \overline{g})$$

AND Gates



$$(h_1 \vee \overline{g})(h_2 \vee \overline{g})(\overline{h}_1 \vee \overline{h}_2 \vee g)$$

OR Gates



$$(\overline{h}_1 \vee g)(\overline{h}_2 \vee g)(h_1 \vee h_2 \vee \overline{g})$$

Output Gate

$$g \bigcirc \text{output} \qquad (g)$$

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- The reduction takes polynomial time

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- In particular, $|S|$ is polynomial in $|I|$

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- Each step of the algorithm $\mathcal{C}(I, S)$ is performed by this computer's circuit
- This gives a circuit of size polynomial in $|I|$ that has input bits for I and S and outputs whether S is a solution for I (a separate circuit for each input length)

Reduction

To solve an instance I of the problem A:

- take a circuit corresponding to $\mathcal{C}(I, \cdot)$

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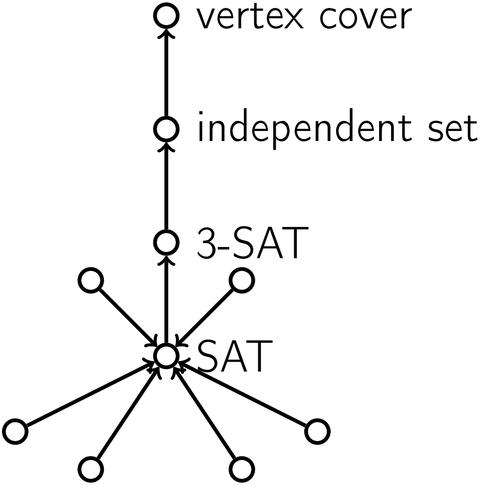
To solve an instance I of the problem A:

- take a circuit corresponding to $\mathcal{C}(I, \cdot)$
- the inputs to this circuit encode candidate solutions
- use a Circuit-SAT algorithm for this circuit to find a solution (if exists)

Summary



Outline

1. Reductions
2. Showing **NP**-completeness
3. Independent Set $\rightarrow$ Vertex Cover
4. 3-SAT $\rightarrow$ Independent Set
5. SAT $\rightarrow$ 3-SAT
6. All of **NP** $\rightarrow$ SAT
7. Using SAT-solvers

Sudoku Puzzle

This part

A simple and efficient Sudoku solver

SAT: Theory and Practice

Theory: we have no algorithm checking the satisfiability of a CNF formula F with n variables in time $\mathrm{poly}(|F|) \cdot 1.99^n$

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Practice: SAT-solvers routinely solve instances with thousands of variables

Solving Hard Problems in Practice

An easy way to solve a hard combinatorial problem in practice:

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An easy way to solve a hard combinatorial problem in practice:

- Reduce the problem to SAT (many problems are reduced to SAT in a natural way)
- Use a SAT solver

Sudoku Puzzle

Goal: fill in with digits the partially completed 9×9 grid so that each row, each column, and each of the nine 3×3 subgrids contains all the digits from 1 to 9.

Example

Variables

There will be $9 \times 9 \times 9 = 729$ Boolean variables: for $1 \leq i, j, k \leq 9$, $x_{ijk} = 1$, if and only if the cell $[i, j]$ contains the digit k

Exactly One Is True

Clauses expressing the fact that exactly one of the literals ℓ_1, ℓ_2, ℓ_3 is equal to 1:

$$(\ell_1 \lor \ell_2 \lor \ell_3)(\overline{\ell}_1 \lor \overline{\ell}_2)(\overline{\ell}_1 \lor \overline{\ell}_3)(\overline{\ell}_2 \lor \overline{\ell}_3)$$

Constraints

- Cell $[i, j]$ contains exactly one digit: $\texttt{ExactlyOneOf}(x_{ij1}, x_{ij2}, \ldots, x_{ij9})$

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- k appears exactly once in a 3×3 block:
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- $[i, j]$ already contains k: (x_{ijk})

Resulting Formula

State-of-the-art SAT-solvers find a satisfying assignment for the resulting formula in blink of an eye, though the corresponding search space has size about $2^{729} \approx 10^{220}$