Advanced Shortest Paths: A-star Algorithm (A^*)

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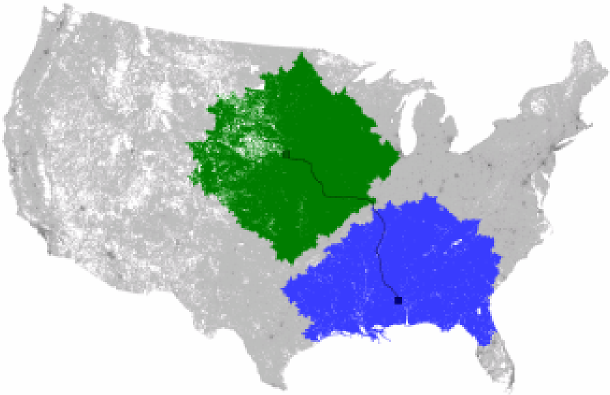
Graph Algorithms
Data Structures and Algorithms

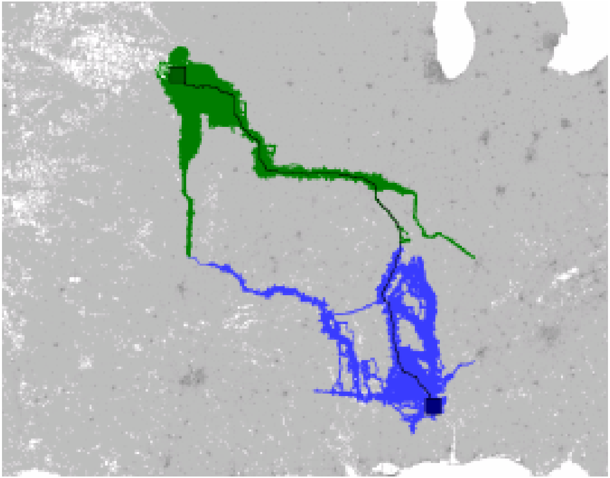
Outline

1. Directed Search
2. Bidirectional A^*
3. Lower Bounds
4. Landmarks

Bidirectional Search



Directed Search



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- It defines new edge weights $\ell_\pi(u, v) = \ell(u, v) - \pi(u) + \pi(v)$
- Replacing ℓ by ℓ_π does not change shortest paths

Lemma

For any potential function $\pi : V \to \mathbb{R}$, for any two vertices s and t in the graph and any path P between them, $\ell_\pi(P) = \ell(P) - \pi(s) + \pi(t)$.

Proof

$P\colon s = v_1 \to v_2 \cdots \to v_k = t$

$$
\begin{aligned}
\ell_\pi(P) &= \sum_{i=1}^{k-1} \ell_\pi(v_i, v_{i+1}) = \\
&= \ell(v_1, v_2) - \pi(v_1) + \pi(v_2) + \\
&+ \ell(v_2, v_3) - \pi(v_2) + \pi(v_3) + \\
&+ \ldots + \\
&+ \ell(v_{k-2}, v_{k-1}) - \pi(v_{k-2}) + \pi(v_{k-1}) + \\
&+ \ell(v_{k-1}, v_k) - \pi(v_{k-1}) + \pi(v_k) = \\
&= \sum_{i=1}^{k-1} \ell(v_i, v_{i+1}) - \pi(v_1) + \pi(v_k) = \\
&= \ell(P) - \pi(s) + \pi(t)
\end{aligned}
$$

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- Does any π fit us?
- For any edge (u, v), the new length $\ell_\pi(u, v)$ must be non-negative — such π is called feasible

Intuition

- $\pi(v)$ is an estimation of $d(v, t)$ — "how far is it from here to t?"
- If we have such estimation, we can often avoid going wrong direction — directed search
- Typically $\pi(v)$ is a lower bound on $d(v, t)$
- I.e., on a real map a path from v to t cannot be shorter than the straight line segment from v to t

$A^* \equiv$ Dijkstra

- On each step, pick the vertex v minimizing $\texttt{dist}[v] - \pi(s) + \pi(v)$
- $\pi(s)$ is the same for all v, so v minimizes $\texttt{dist}[v] + \pi(v)$ — the most promising vertex
- $\pi(v)$ is an estimate of $d(v, t)$
- Pick the vertex v with the minimum current estimate of $d(s, v) + d(v, t)$
- Thus the search is directed

Performance of A^*

If $\pi(v)$ gives lower bound on $d(v, t)$

- Worst case: $\pi(v) = 0$ for all v — the same as Dijkstra
- Best case: $\pi(v) = d(v, t)$ for all v — then $\ell_\pi(u, v) = 0$ iff (u, v) is on a shortest path to t, so search visits only the edges of shortest $s - t$ paths
- It can be shown that the tighter are the lower bounds — the fewer vertices will be scanned

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Bidirectional A^*

- Same as Bidirectional Dijkstra, but with potentials
- Needs two potential functions: $\pi_f(v)$ estimates $d(v, t)$, $\pi_r(v)$ estimates $d(s, v)$
- Problem: different edge weights: $\ell_{\pi_f}(u, v) = \ell(u, v) - \pi_f(u) + \pi_f(v)$, $\ell_{\pi_r}(u, v) = \ell(u, v) - \pi_r(v) + \pi_r(u)$

Bidirectional A^*

- We need $\ell_{\pi_f}(u, v) = \ell_{\pi_r}(u, v) \Rightarrow \pi_f(u) + \pi_r(u) = \pi_f(v) + \pi_r(v)$ for any (u, v)
- Need constant $\pi_f(u) + \pi_r(u)$ for any u
- Use $p_f(u) = \frac{\pi_f(u) - \pi_r(u)}{2}$, $p_r(u) = -p_f(u)$
- Then $p_f(u) + p_r(u) = 0$ for any u

Lemma

If π_f is a feasible potential for forward search, and π_r is a feasible potential for reverse search, then $p_f = \frac{\pi_f - \pi_r}{2}$ is a feasible potential for forward search.

Proof

- $\ell(u, v) - \pi_f(u) + \pi_f(v) \geq 0$

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- $\ell(u, v) - \pi_r(v) + \pi_r(u) \geq 0$
- $2\ell(u, v) - (\pi_f(u) - \pi_r(u)) + (\pi_f(v) - \pi_r(v)) \geq 0$

Proof

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- $2\ell(u, v) - (\pi_f(u) - \pi_r(u)) + (\pi_f(v) - \pi_r(v)) \geq 0$
- $\ell(u, v) - \frac{\pi_f(u) - \pi_r(u)}{2} + \frac{\pi_f(v) - \pi_r(v)}{2} \geq 0$

Proof

- $\ell(u,v) - \pi_f(u) + \pi_f(v) \geq 0$
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- $2\ell(u,v) - (\pi_f(u) - \pi_r(u)) + (\pi_f(v) - \pi_r(v)) \geq 0$
- $\ell(u,v) - \frac{\pi_f(u)-\pi_r(u)}{2} + \frac{\pi_f(v)-\pi_r(v)}{2} \geq 0$
- $\ell(u,v) - p_f(u) + p_f(v) \geq 0$ $\square$

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Lemma

If π is feasible, and $\pi(t) \leq 0$, then $\pi(v) \leq d(v, t)$ for any v

Proof

- $\ell_\pi(x, y) \geq 0$ for any x, y, so $\ell_\pi(P) \geq 0$ for any path P
- Take a $v - t$ shortest path $P = (v, w_1, w_2, \ldots, w_k, t)$
- $0 \leq \ell_\pi(P) = \ell(P) - \pi(v) + \pi(t) \leq \ell(P) - \pi(v) \Rightarrow \pi(v) \leq \ell(P) = d(v, t)$ $\square$

Euclidean Potential

Lemma

Consider a road network on a plane map with each vertex v having coordinates $(v.x, v.y)$. The potential given by Euclidean distance (length of a line segment) between v and t $\pi(v) = d_E(v, t) = \sqrt{(v.x - t.x)^2 + (v.y - t.y)^2}$ is feasible, and $\pi(t) = 0$.

Proof

- For any edge $(u, v) \in E$, $\ell(u, v) \geq d_E(u, v)$, because line segment is the shortest path between two points on a plane

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- $\pi(u) = d_E(u, t) \leq_{\texttt{(triangle inequality)}} d_E(u, v) + d_E(v, t) \leq \ell(u, v) + \pi(v) \Rightarrow \ell(u, v) - \pi(u) + \pi(v) \geq 0$

Proof

- For any edge $(u, v) \in E$, $\ell(u, v) \geq d_E(u, v)$, because line segment is the shortest path between two points on a plane
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- $\pi(t) = d_E(t, t) = 0$ $\square$

A^* on a Plane Map

- Need to find the shortest path from s to t
- For each v, compute $\pi(v) = d_E(v, t)$
- Launch Dijkstra with potentials $\pi(v)$

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Landmarks

Lemma

Fix some vertex $A \in V$, we will call it a landmark. Then the potential $\pi(v) = d(A, t) - d(A, v)$ is feasible, and $\pi(t) = 0$.

Proof

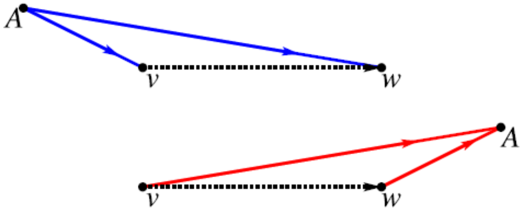
- $\ell(u,v) - \pi(u) + \pi(v) = \ell(u,v) - d(A,t) + d(A,u) + d(A,t) - d(A,v) = d(A,u) + \ell(u,v) - d(A,v) \geq_{\texttt{(triangle inequality)}} 0$
- $\pi(t) = d(A,t) - d(A,t) = 0$ $\square$

Landmarks

- Select several landmarks and precompute their distances to all other vertices
- For any landmark A,
  $d(v, t) \geq d(A, t) - d(A, v)$,
  $d(v, t) \geq d(v, A) - d(t, A)$
- Tightest lower bound $d(v, t) \geq \max(d(A, t) - d(A, v), d(v, A) - d(t, A))$ over all A

Landmark Selection

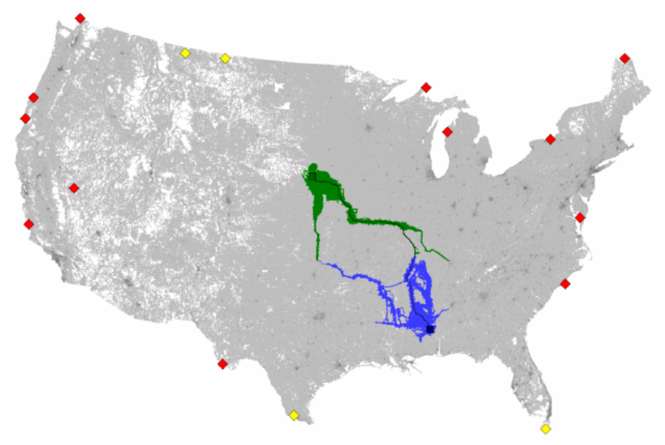
Good landmark appears "before" v or "after" w:



For any query (s, t), we need some landmarks before s and after t

Landmark Selection

Choosing landmarks on the border seems reasonable:



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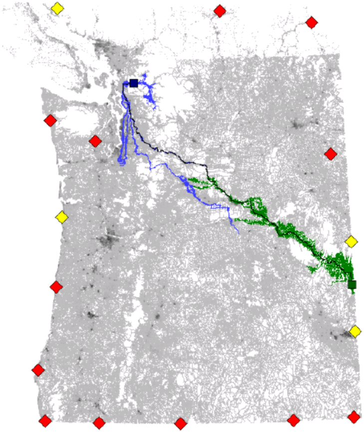

Conclusion

- Directed search can scan fewer vertices
- A^* is a directed search algorithm based on Dijkstra and potential functions
- A^* can also be bidirectional
- Euclidean distance is a potential for a plane (road networks)
- Landmarks can be used for good potential function, but we need preprocessing to use them