Algorithmic Challenges: Knuth-Morris-Pratt Algorithm

Michael Levin

Higher School of Economics

Algorithms on Strings
Data Structures and Algorithms

Outline

1. Exact Pattern Matching
2. Safe Shift
3. Prefix Function
4. Computing Prefix Function
5. Knuth-Morris-Pratt Algorithm

Exact Pattern Matching

Input: Strings T (Text) and P (Pattern).

Output: All such positions in T (Text) where P (Pattern) appears as a substring.

(For all strings in this module we use 0-based indices)

Brute Force Algorithm

- Slide the Pattern down Text

Brute Force Algorithm

- Slide the Pattern down Text
- Running time $\Theta(|T||P|)$

Brute Force Algorithm

a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							

Output: []

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
	a	b	r	a						

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
	a	b	r	a						

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
		a	b	r	a					

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
		a	b	r	a					

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
			a	b	r	a				

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
			a	b	r	a				

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
				a	b	r	a			

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
				a	b	r	a			

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
					a	b	r	a		

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
					a	b	r	a		

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a

a	b	r	a

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
						a	b	r	a	

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
							a	b	r	a

Output: [0]

Brute Force Algorithm

0	1	2	3	4	5	6	7	8	9	10
a	b	r	a	c	a	d	a	b	r	a
							a	b	r	a

Output: [0,7]

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a

a	b	r	a

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a

a	b	r	a

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							
	a	b	r	a						

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							
	a	b	r	a						

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							
	a	b	r	a						

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							
	a	b	r	a						

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							
		a	b	r	a					

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							
			a	b	r	a				

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a
a	b	r	a							
			a	b	r	a				

Skipping Positions

a	b	r	a	c	a	d	a	b	r	a
			a	b	r	a				
			a	b	r	a				

Skipping Positions

a	b	c	d	a	b	c	d	a	b	e	f

a	b	c	d	a	b	e	f

Skipping Positions

a	b	c	d	a	b	c	d	a	b	e	f

a	b	c	d	a	b	e	f

Skipping Positions

a	b	c	d	a	b	c	d	a	b	e	f

a	b	c	d	a	b	e	f

Skipping Positions

a	b	c	d	a	b	c	d	a	b	e	f
				a	b	c	d	a	b	e	f

Skipping Positions

a	b	c	d	a	b	c	d	a	b	e	f
				a	b	c	d	a	b	e	f

Skipping Positions

a	b	a	b	a	b	a	b	a	b	e	f

a	b	a	b	a	b	e	f

Skipping Positions

a	b	a	b	a	b	a	b	a	b	e	f

a	b	a	b	a	b	e	f

Skipping Positions

a	b	a	b	a	b	a	b	a	b	e	f

a	b	a	b	a	b	e	f

Skipping Positions

a	b	a	b	a	b	a	b	a	b	e	f

a	b	a	b	a	b	e	f

Skipping Positions

a	b	a	b	a	b	a	b	a	b	e	f

a	b	a	b	a	b	e	f

Skipping Positions

a	b	a	b	a	b	a	b	a	b	e	f

a	b	a	b	a	b	e	f

Skipping Positions

a	b	a	b	a	b	a	b	a	b	e	f
				a	b	a	b	a	b	e	f

Skipping Positions

a	b	a	b	a	b	a	b	a	b	e	f
				a	b	a	b	a	b	e	f

Definitions

Definition

Border of string S is a prefix of S which is equal to a suffix of S, but not equal to the whole S.

Example

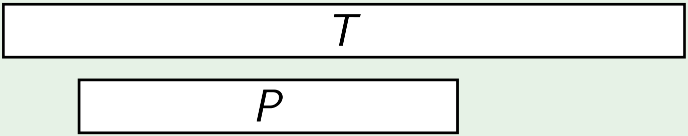
"a" is a border of "arba"

"ab" is a border of "abcdab"

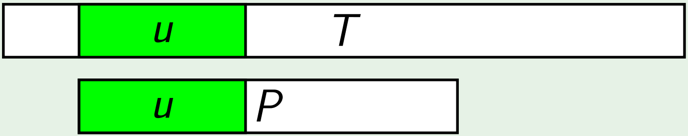
"abab" is a border of "ababab"

"ab" is not a border of "ab"

Shifting Pattern

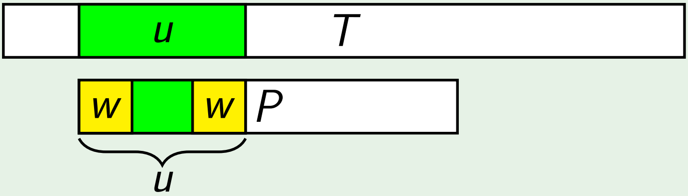


Shifting Pattern



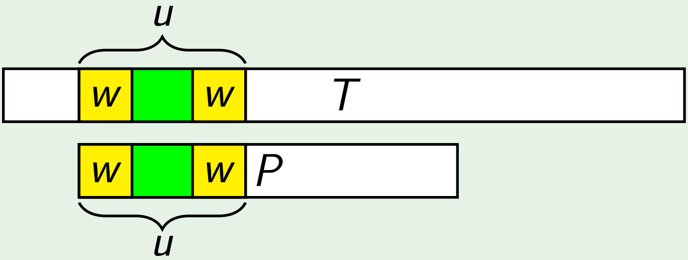
- Find longest common prefix u

Shifting Pattern



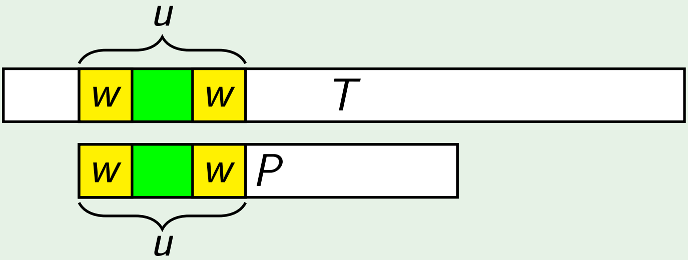
- Find longest common prefix u
- Find w — the longest border of u

Shifting Pattern



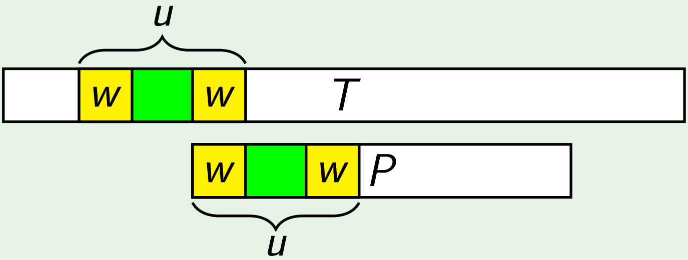
- Find longest common prefix u
- Find w — the longest border of u

Shifting Pattern



- Find longest common prefix u
- Find w — the longest border of u
- Move P such that prefix w in P aligns with suffix w of u in T

Shifting Pattern



- Find longest common prefix u
- Find w — the longest border of u
- Move P such that prefix w in P aligns with suffix w of u in T

- Now you know we can skip some of the comparisons

- Now you know we can skip some of the comparisons
- But we shouldn't miss any of the pattern occurrences in the text

- Now you know we can skip some of the comparisons
- But we shouldn't miss any of the pattern occurrences in the text
- Is it safe to shift the pattern this way?

Outline

Suffix notation

Definition

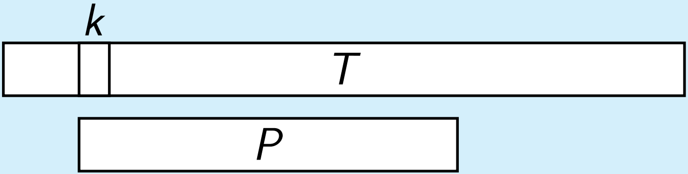
Denote by S_k suffix of string S starting at position k.

Examples

S = "abcd" $\Rightarrow$ S_2 = "cd"
T = "abc" $\Rightarrow$ T_0 = "abc"
P = "aa" $\Rightarrow$ P_1 = "a"

Safe shift

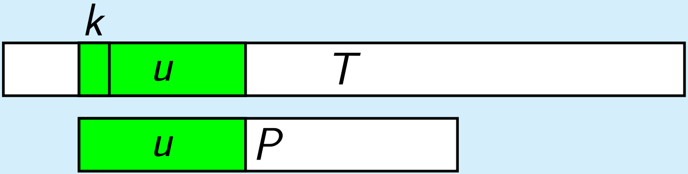
Lemma



Let u be the longest common prefix of P and T_k. Let w be the longest border of u. Then there are no occurrences of P in T starting between positions k and $(k + |u| - |w|)$ — the start of suffix w in the prefix u of T_k.

Safe shift

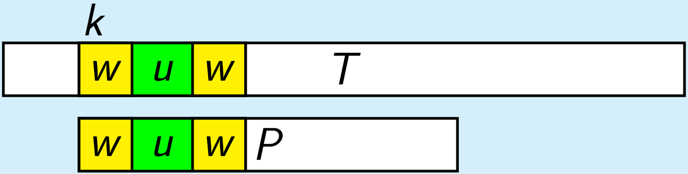
Lemma



Let u be the longest common prefix of P and T_k. Let w be the longest border of u. Then there are no occurrences of P in T starting between positions k and $(k + |u| - |w|)$ — the start of suffix w in the prefix u of T_k.

Safe shift

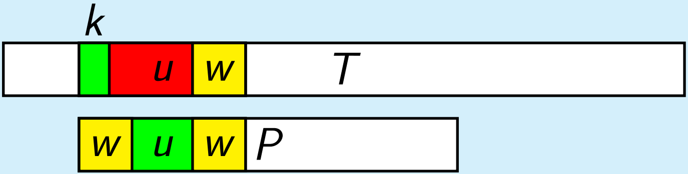
Lemma



Let u be the longest common prefix of P and T_k. Let w be the longest border of u. Then there are no occurrences of P in T starting between positions k and $(k + |u| - |w|)$ — the start of suffix w in the prefix u of T_k.

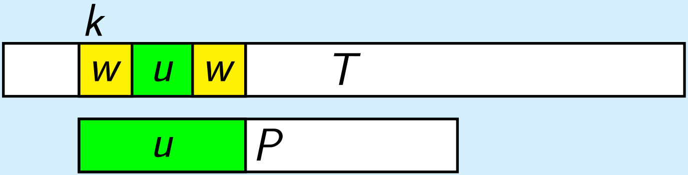
Safe shift

Lemma

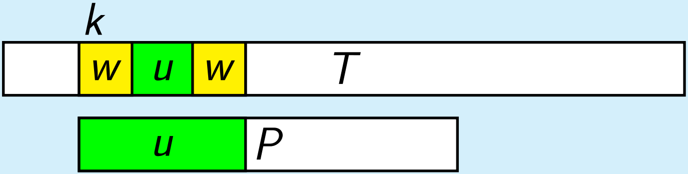


Let u be the longest common prefix of P and T_k. Let w be the longest border of u. Then there are no occurrences of P in T starting between positions k and $(k + |u| - |w|)$ — the start of suffix w in the prefix u of T_k.

Proof

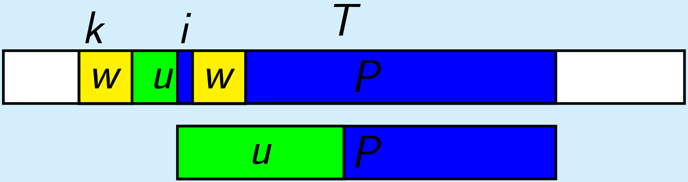


Proof



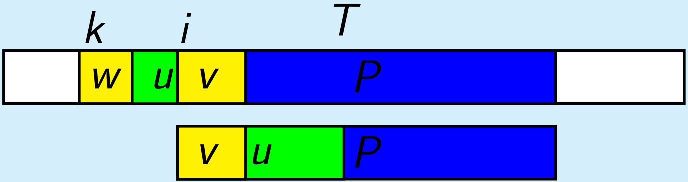
- Suppose P occurs in T in position i between k and start of suffix w

Proof



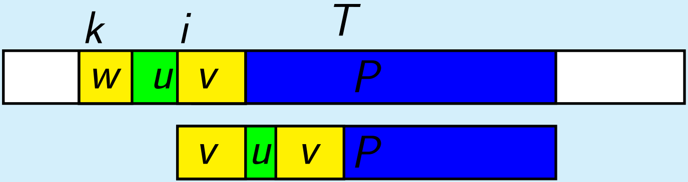
- Suppose P occurs in T in position i between k and start of suffix w

Proof



- Suppose P occurs in T in position i between k and start of suffix w
- Then there is prefix v of P equal to suffix in u, and v is longer than w $\square$

Proof



- Then there is prefix v of P equal to suffix in u, and v is longer than w
- v is a border longer than w, but w is longest border of u — contradiction □

- Now you know it is possible to avoid many of the comparisons which Brute Force algorithm does

- Now you know it is possible to avoid many of the comparisons which Brute Force algorithm does
- But how to determine the best pattern shifts?

Outline

1. Exact Pattern Matching
2. Safe Shift
3. Prefix Function
4. Computing Prefix Function
5. Knuth-Morris-Pratt Algorithm

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Prefix function

Definition

Prefix function of a string P is a function $s(i)$ that for each i returns the length of the longest border of the prefix $P[0..i]$.

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Lemma

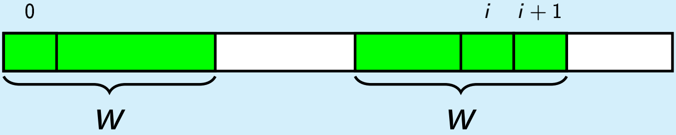
$P[0..i]$ has a border of length $s(i+1)-1$

Proof

Lemma

$P[0..i]$ has a border of length $s(i+1)-1$

Proof

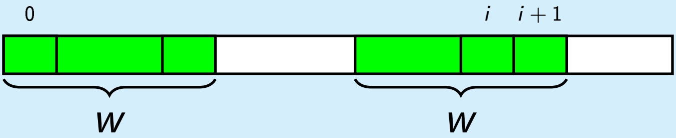


- Take the longest border w of $P[0..i+1]$

Lemma

$P[0..i]$ has a border of length $s(i+1)-1$

Proof

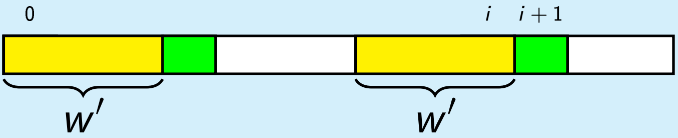


- Take the longest border w of $P[0..i+1]$
- Cut the last character from w — it's a border of $P[0..i]$ now $\square$

Lemma

$P[0..i]$ has a border of length $s(i+1)-1$

Proof



- Take the longest border w of $P[0..i+1]$
- Cut the last character from w — it's a border of $P[0..i]$ now $\square$

Corollary

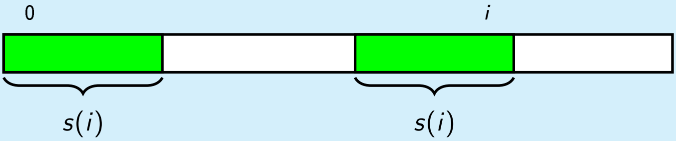
$$s(i+1) \leq s(i) + 1$$

Enumerating borders

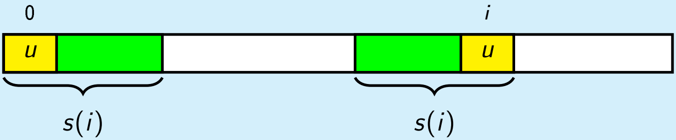
Lemma

If $s(i) > 0$, then all borders of $P[0..i]$ but for the longest one are also borders of $P[0..s(i)-1]$.

Proof

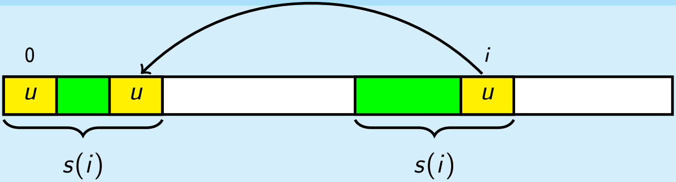


Proof



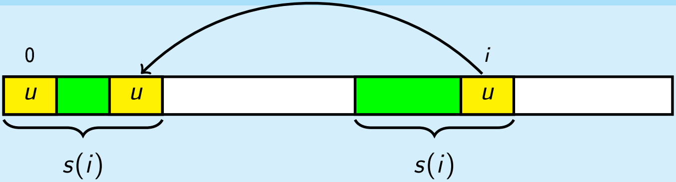
- Let u be a border of $P[0..i]$ such that $|u| < s(i)$

Proof



- Let u be a border of $P[0..i]$ such that $|u| < s(i)$
- Then u is both a prefix and a suffix of $P[0..s(i) - 1]$

Proof



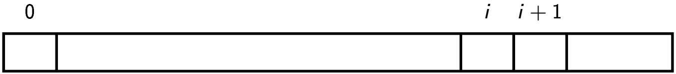
- Let u be a border of $P[0..i]$ such that $|u| < s(i)$
- Then u is both a prefix and a suffix of $P[0..s(i)-1]$
- $u \neq P[0..s(i)-1]$, so u is a border of $P[0..s(i)-1]$

Enumerating borders

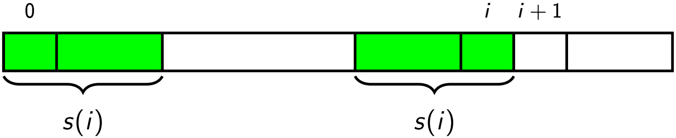
Corollary

All borders of $P[0..i]$ can be enumerated by taking the longest border b_1 of $P[0..i]$, then the longest border b_2 of b_1, then the longest border b_3 of b_2, $\ldots$, and so on.

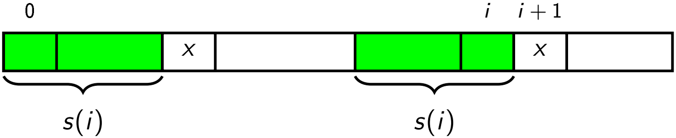
Computing $s(i+1)$



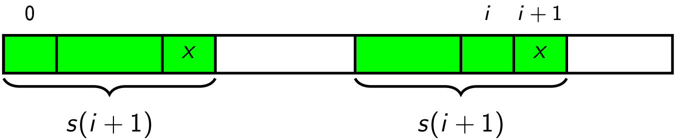
Computing $s(i+1)$



Computing $\boldsymbol{s(i+1)}$

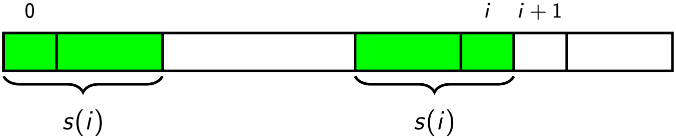


Computing $s(i+1)$

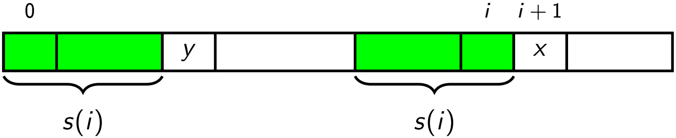


$$s(i+1) = s(i) + 1$$

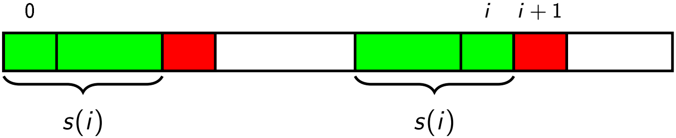
Computing $s(i+1)$



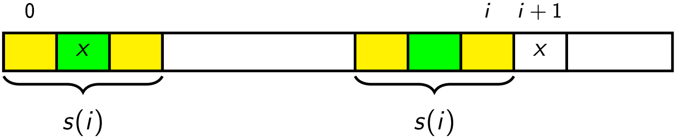
Computing $s(i+1)$



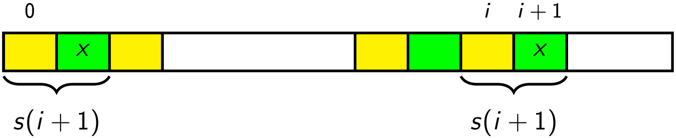
Computing $s(i+1)$



Computing $s(i+1)$



Computing $s(i+1)$



$$s(i+1) = |some\ border\ of\ P[0..s(i)-1]| + 1$$

- Now you know lots of properties of prefix function

- Now you know lots of properties of prefix function
- But how to compute all of its values??

Outline

Example

P	a	b	a	b	a	b	c	a	a	b
s										

Example

P	a	b	a	b	a	b	c	a	a	b
s										

Example

P	a	b	a	b	a	b	c	a	a	b
s	0									

Example

P	a	b	a	b	a	b	c	a	a	b
s	0									

Example

P	a	b	a	b	a	b	c	a	a	b
s	0									

Example

P	a	b	a	b	a	b	c	a	a	b
s	0									

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0								

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0								

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0								

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1							

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1							

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1							

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2						

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2						

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2						

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3					

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3					

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3					

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4				

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4				

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4				

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4				

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4				

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4				

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4				

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0			

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0			

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0			

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1		

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1		

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1		

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1		

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1		

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

Example

P	a	b	a	b	a	b	c	a	a	b
s	0	0	1	2	3	4	0	1	1	2

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
  while (border > 0) and (P[i] ≠ P[border]):
    border ← s[border − 1]
  if P[i] == P[border]:
    border ← border + 1
  else:
    border ← 0
  s[i] ← border
return s
```

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
  while (border > 0) and (P[i] ≠ P[border]):
    border ← s[border − 1]
  if P[i] == P[border]:
    border ← border + 1
  else:
    border ← 0
  s[i] ← border
return s
```

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
  while (border > 0) and (P[i] ≠ P[border]):
    border ← s[border − 1]
  if P[i] == P[border]:
    border ← border + 1
  else:
    border ← 0
  s[i] ← border
return s
```

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
  while (border > 0) and (P[i] ≠ P[border]):
    border ← s[border − 1]
  if P[i] == P[border]:
    border ← border + 1
  else:
    border ← 0
  s[i] ← border
return s
```

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
    while (border > 0) and (P[i] ≠ P[border]):
        border ← s[border − 1]
    if P[i] == P[border]:
        border ← border + 1
    else:
        border ← 0
    s[i] ← border
return s
```

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
    while (border > 0) and (P[i] ≠ P[border]):
        border ← s[border − 1]
    if P[i] == P[border]:
        border ← border + 1
    else:
        border ← 0
    s[i] ← border
return s
```

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
    while (border > 0) and (P[i] ≠ P[border]):
        border ← s[border − 1]
    if P[i] == P[border]:
        border ← border + 1
    else:
        border ← 0
    s[i] ← border
return s
```

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
    while (border > 0) and (P[i] ≠ P[border]):
        border ← s[border − 1]
    if P[i] == P[border]:
        border ← border + 1
    else:
        border ← 0
    s[i] ← border
return s
```

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
    while (border > 0) and (P[i] ≠ P[border]):
        border ← s[border − 1]
    if P[i] == P[border]:
        border ← border + 1
    else:
        border ← 0
    s[i] ← border
return s
```

```
ComputePrefixFunction(P)
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
  while (border > 0) and (P[i] ≠ P[border]):
    border ← s[border − 1]
  if P[i] == P[border]:
    border ← border + 1
  else:
    border ← 0
  s[i] ← border
return s
```

ComputePrefixFunction(P)

```
s ← array of integers of length |P|
s[0] ← 0, border ← 0
for i from 1 to |P| − 1:
  while (border > 0) and (P[i] ≠ P[border]):
    border ← s[border − 1]
  if P[i] == P[border]:
    border ← border + 1
  else:
    border ← 0
  s[i] ← border
return s
```

Lemma

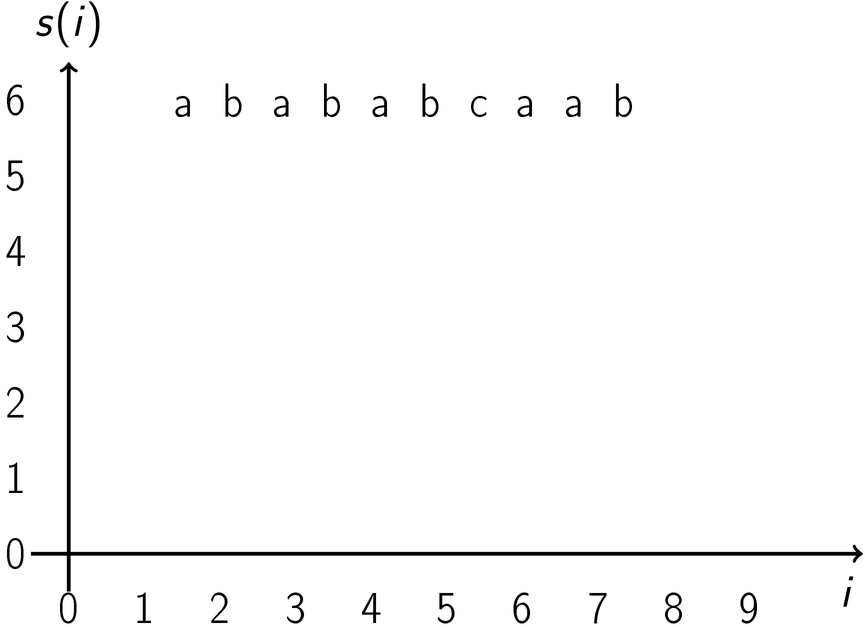
The running time of `ComputePrefixFunction` is $O(|P|)$.

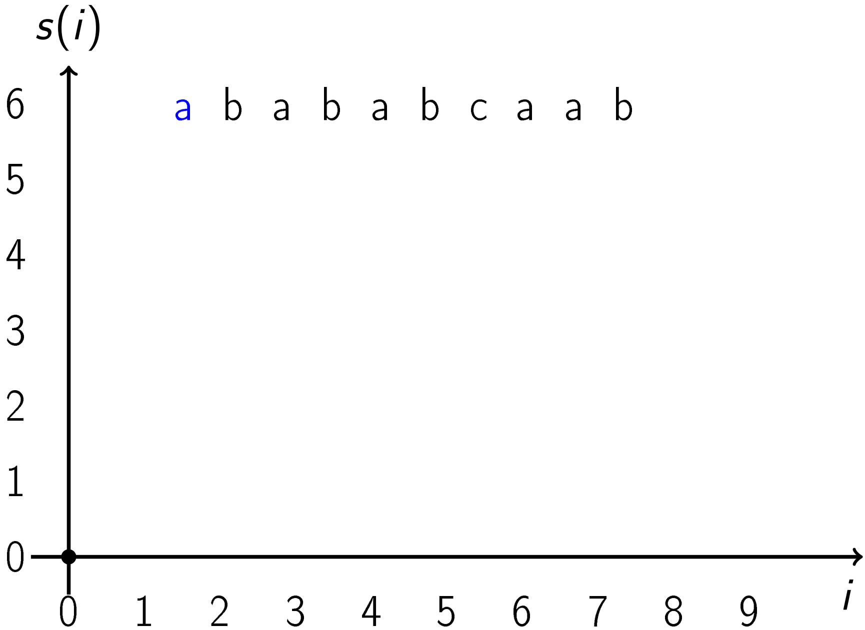
Proof

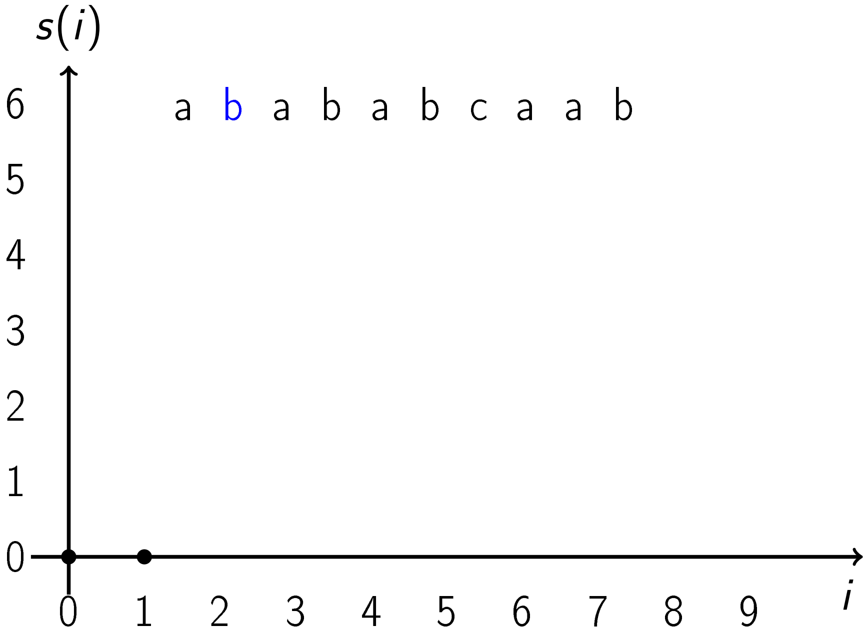
- Everything but for inner while loop is $O(|P|)$ initialization plus $O(|P|)$ iterations of the for loop with $O(1)$ assignments on each iteration

Proof

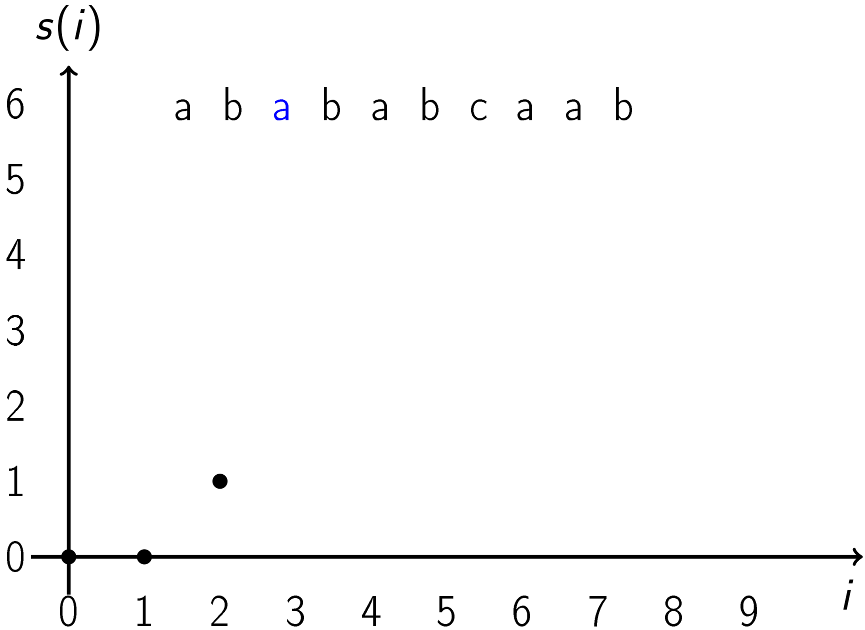
- Everything but for inner while loop is $O(|P|)$ initialization plus $O(|P|)$ iterations of the for loop with $O(1)$ assignments on each iteration
- Now we will bound the number of the while loop iterations by $O(|P|)$

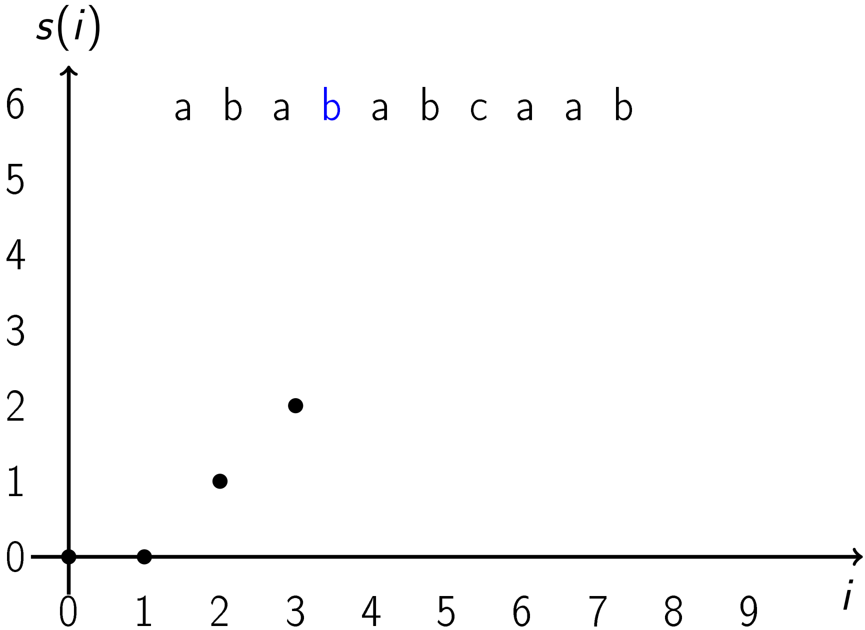

s(i)
6
5
4
3
2
1
0
a b a b a b c a a b
0 1 2 3 4 5 6 7 8 9
i


s(i)
6
5
4
3
2
1
0
a b a b a b c a a b
0 1 2 3 4 5 6 7 8 9
i

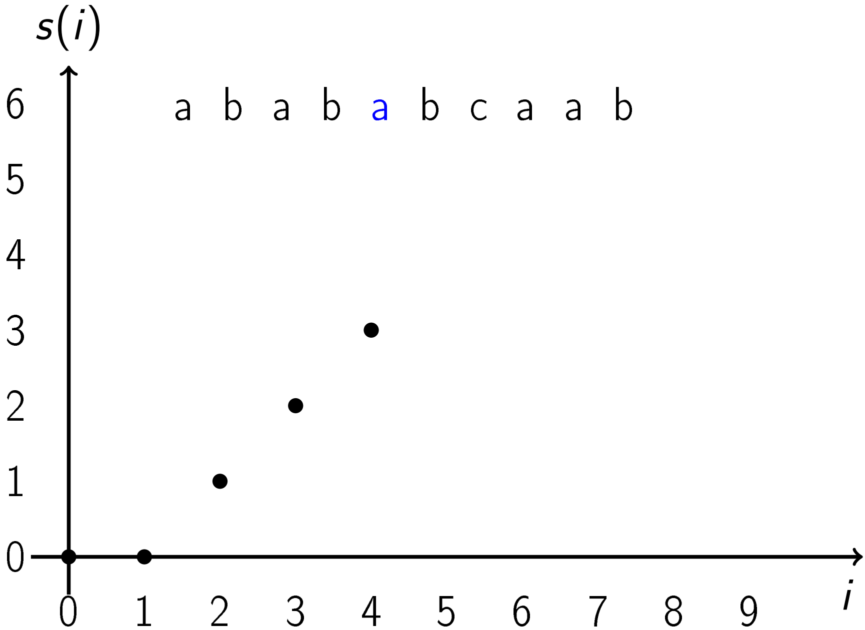


$s(i)$
a b a b a b c a a b
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9
i

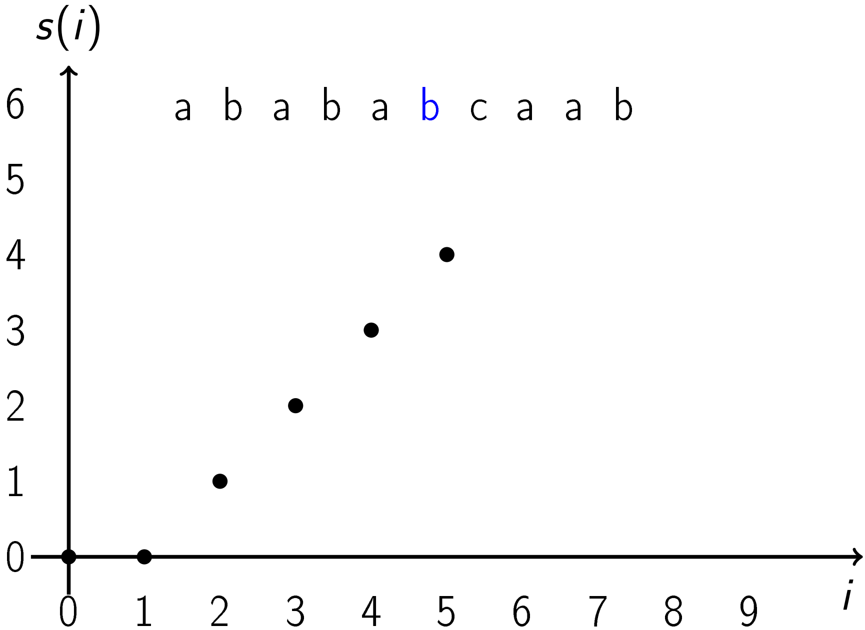

s(i)
a b a b a b c a a b
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9
i



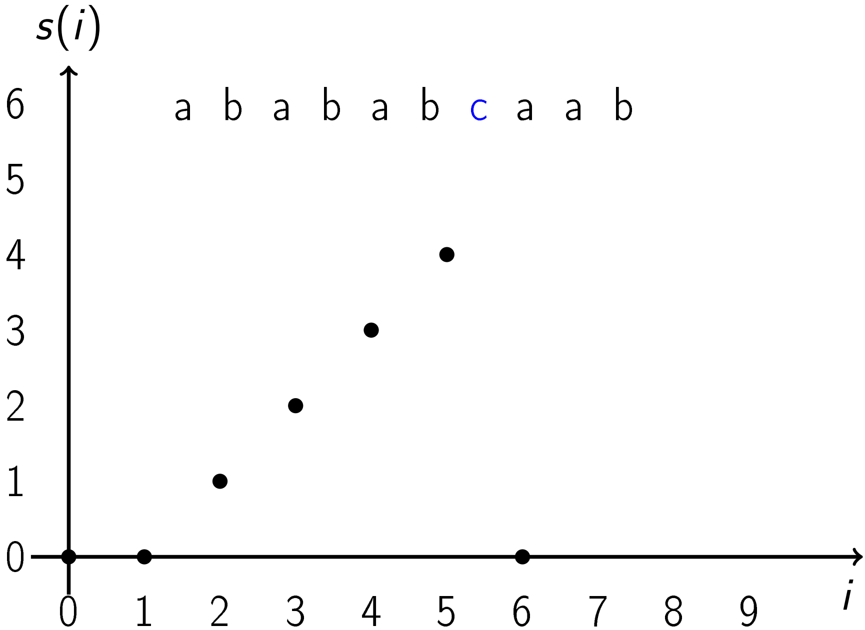
$s(i)$
a b a b a b c a a b
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9
i

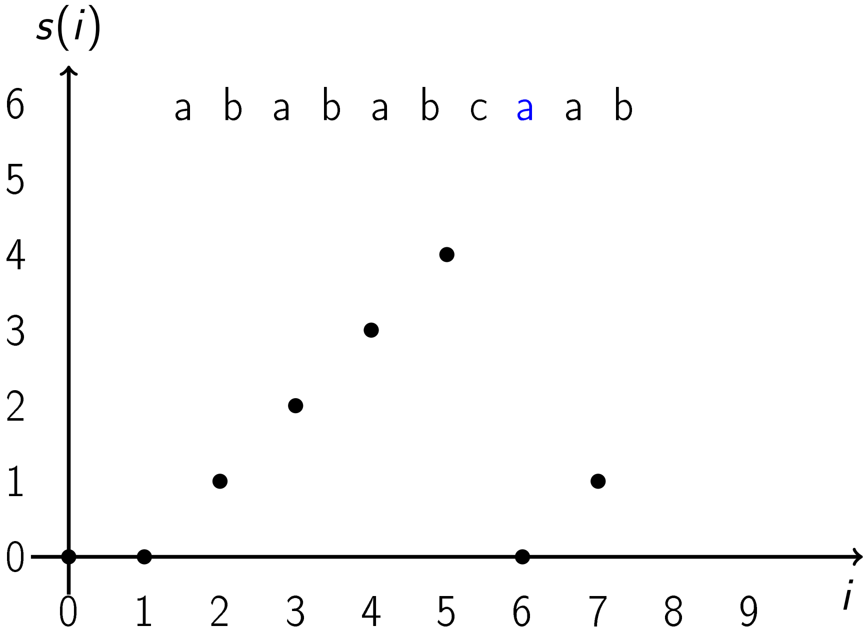


s(i)
a b a b a b c a a b
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9
i

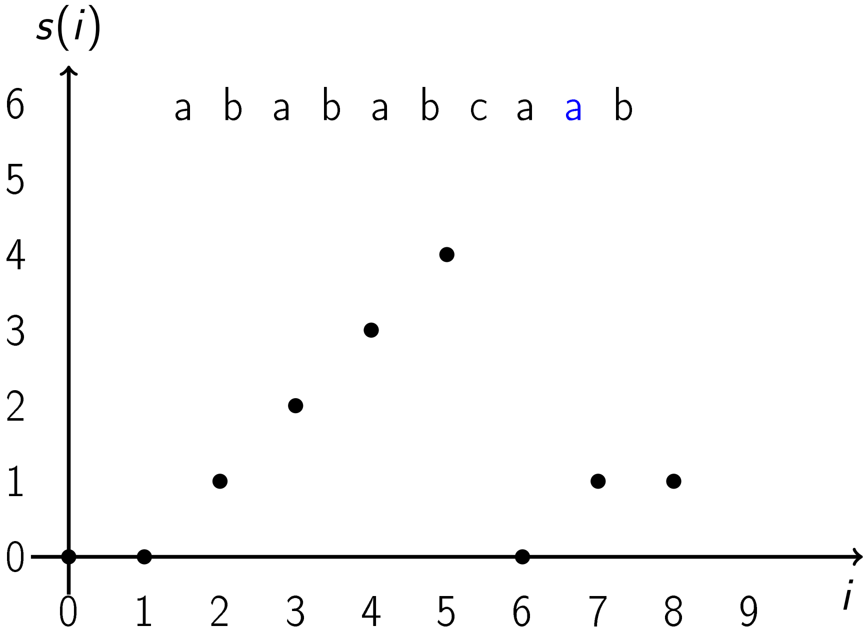


s(i)
a b a b a b c a a b
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9
i

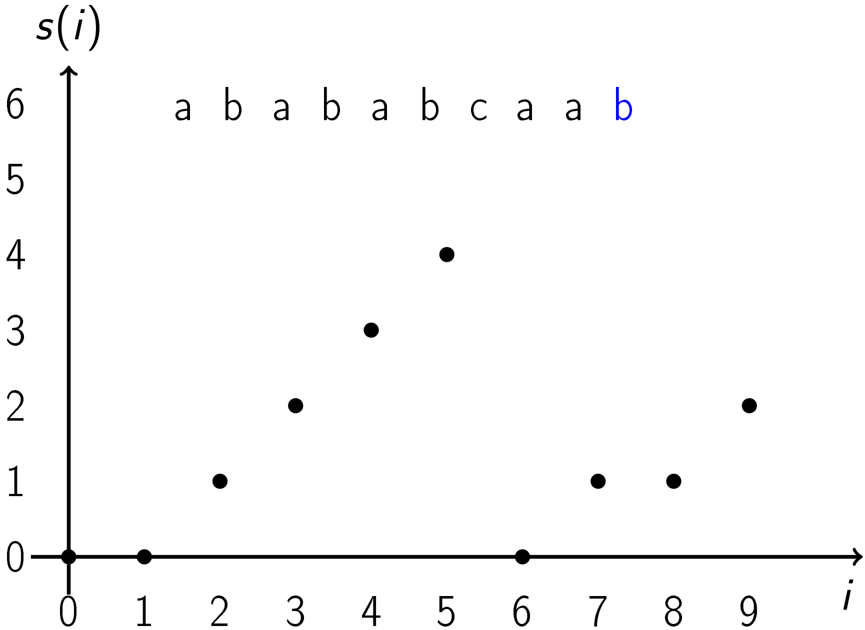

$s(i)$
a b a b a b c a a b
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9
i



s(i)
a b a b a b c a a b
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9
i



s(i)
6
5
4
3
2
1
0
a b a b a b c a a b
0 1 2 3 4 5 6 7 8 9
i


s(i)
a b a b a b c a a b
0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9
i

Proof

(continued)

- *border* can increase at most by 1 on each iteration of the for loop

Proof

(continued)

- *border* can increase at most by 1 on each iteration of the for loop
- In total, *border* is increased $O(|P|)$ times

Proof

(continued)

- *border* can increase at most by 1 on each iteration of the for loop
- In total, *border* is increased $O(|P|)$ times
- *border* is decreased at least by 1 on each iteration of the while loop

Proof

(continued)

- *border* can increase at most by 1 on each iteration of the for loop
- In total, *border* is increased $O(|P|)$ times
- *border* is decreased at least by 1 on each iteration of the while loop
- *border* ≥ 0

Proof

(continued)

- *border* can increase at most by 1 on each iteration of the for loop
- In total, *border* is increased $O(|P|)$ times
- *border* is decreased at least by 1 on each iteration of the while loop
- *border* ≥ 0
- So there are $O(|P|)$ iterations of the while loop $\square$

- Now you know how to compute prefix function in linear time

- Now you know how to compute prefix function in linear time
- But how to find pattern in text??

Outline

Algorithm

			P									T					
S	a	b	r	a	$	a	b	r	a	c	a	d	a	b	r	a

To search for pattern P in text T:

- Create new string $S = P +$ '\$' $+ T$, where '\$' is a special character absent from both P and T

Algorithm

	P								T							
S	a	b	r	a	\$	a	b	r	a	c	a	d	a	b	r	a
s	0	0	0	1	0	1	2	3	4	0	1	0	1	2	3	4

To search for pattern P in text T:

- Compute prefix function s for string S

Algorithm

	P								T							
S	a	b	r	a	\$	a	b	r	a	c	a	d	a	b	r	a
s	0	0	0	1	0	1	2	3	4	0	1	0	1	2	3	4

To search for pattern P in text T:

- Compute prefix function s for string S
- For all positions i such that $i > |P|$ and $s(i) = |P|$, add $i - 2|P|$ to the output

Algorithm

	P								T							
S	a	b	r	a	\$	a	b	r	a	c	a	d	a	b	r	a
s	0	0	0	1	0	1	2	3	4	0	1	0	1	2	3	4

To search for pattern P in text T:

- Compute prefix function s for string S
- For all positions i such that $i > |P|$ and $s(i) = |P|$, add $i - 2|P|$ to the output

Algorithm

	P								T							
S	a	b	r	a	\$	a	b	r	a	c	a	d	a	b	r	a
s	0	0	0	1	0	1	2	3	4	0	1	0	1	2	3	4

To search for pattern P in text T:

- Compute prefix function s for string S
- For all positions i such that $i > |P|$ and $s(i) = |P|$, add $i - 2|P|$ to the output

Algorithm

	P								T							
S	a	b	r	a	\$	a	b	r	a	c	a	d	a	b	r	a
s	0	0	0	1	0	1	2	3	4	0	1	0	1	2	3	4

To search for pattern P in text T:

- Compute prefix function s for string S
- For all positions i such that $i > |P|$ and $s(i) = |P|$, add $i - 2|P|$ to the output

Algorithm

	P							T								
S	a	b	r	a	\$	a	b	r	a	c	a	d	a	b	r	a
s	0	0	0	1	0	1	2	3	4	0	1	0	1	2	3	4

To search for pattern P in text T:

- Compute prefix function s for string S
- For all positions i such that $i > |P|$ and $s(i) = |P|$, add $i - 2|P|$ to the output

Explanation

- For all i, $s(i) \leq |P|$ because of the special character '\$'
- If $i > |P|$ and $s(i) = |P|$, then $P = S[0..|P|-1] = S[i-|P|+1..i] = T[i-2|P|..i-|P|-1]$
- If $s(i) < |P|$, no full occurrence of $|P|$ ends in position i

```
FindAllOccurrences(P, T)
S ← P + '$' + T
s ← ComputePrefixFunction(S)
result ← empty list
for i from |P| + 1 to |S| − 1:
  if s[i] == |P|:
    result.Append(i − 2|P|)
return result
```

```
FindAllOccurrences(P, T)
S ← P + '$' + T
s ← ComputePrefixFunction(S)
result ← empty list
for i from |P| + 1 to |S| − 1:
    if s[i] == |P|:
        result.Append(i − 2|P|)
return result
```

`FindAllOccurrences`(P, T)

$S \leftarrow P +$ '\$' $+ T$
$s \leftarrow$ `ComputePrefixFunction(S)`
`result` $\leftarrow$ `empty list`
`for` i `from` $|P| + 1$ `to` $|S| - 1$:
  `if` $s[i] == |P|$:
    `result.Append`$(i - 2|P|)$
`return result`

FindAllOccurrences(P, T)

```
S ← P + '$' + T
s ← ComputePrefixFunction(S)
result ← empty list
for i from |P| + 1 to |S| − 1:
  if s[i] == |P|:
    result.Append(i − 2|P|)
return result
```

```
FindAllOccurrences(P, T)

S ← P + '$' + T
s ← ComputePrefixFunction(S)
result ← empty list
for i from |P| + 1 to |S| − 1:
  if s[i] == |P|:
    result.Append(i − 2|P|)
return result
```

```
FindAllOccurrences(P, T)
S ← P + '$' + T
s ← ComputePrefixFunction(S)
result ← empty list
for i from |P| + 1 to |S| − 1:
  if s[i] == |P|:
    result.Append(i − 2|P|)
return result
```

```
FindAllOccurrences(P, T)
S ← P + '$' + T
s ← ComputePrefixFunction(S)
result ← empty list
for i from |P| + 1 to |S| − 1:
  if s[i] == |P|:
    result.Append(i − 2|P|)
return result
```

Lemma

The running time of Knuth-Morris-Pratt algorithm is $O(|P| + |T|)$.

Proof

- Building string S is $O(|P| + |T|)$

Lemma

The running time of Knuth-Morris-Pratt algorithm is $O(|P| + |T|)$.

Proof

- Building string S is $O(|P| + |T|)$
- Computing prefix function is $O(|S|) = O(|P| + |T|)$

Lemma

The running time of Knuth-Morris-Pratt algorithm is $O(|P| + |T|)$.

Proof

- Building string S is $O(|P| + |T|)$
- Computing prefix function is $O(|S|) = O(|P| + |T|)$
- The for loop runs $O(|S|) = O(|P| + |T|)$ iterations $\square$

Conclusion

- Can search pattern in text in linear time
- Can compute prefix function of a string in linear time
- Can enumerate all borders of a string