Disjoint Sets: Naive Implementations

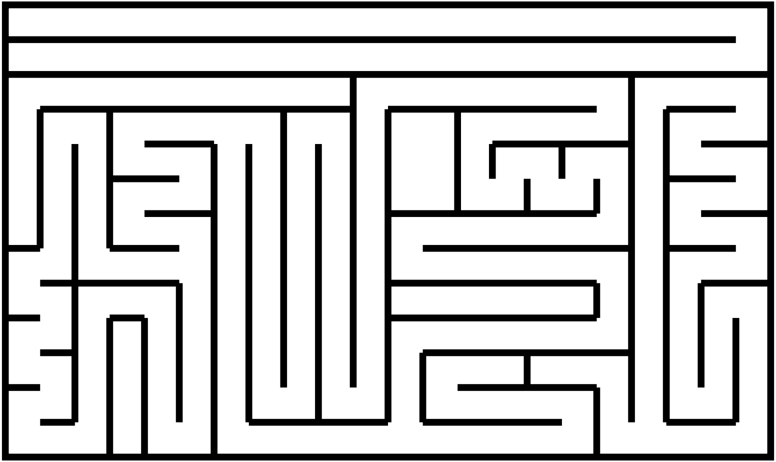
Alexander S. Kulikov

Steklov Institute of Mathematics at St. Petersburg
Russian Academy of Sciences

Data Structures Fundamentals
Algorithms and Data Structures

Outline

Maze: Is B Reachable from A?



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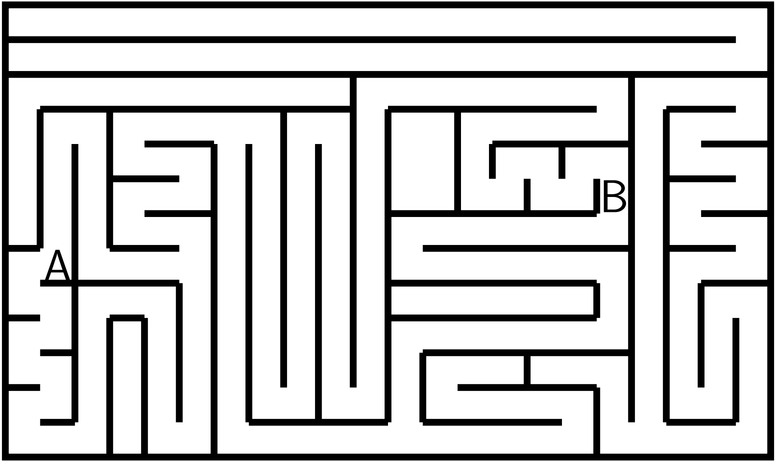

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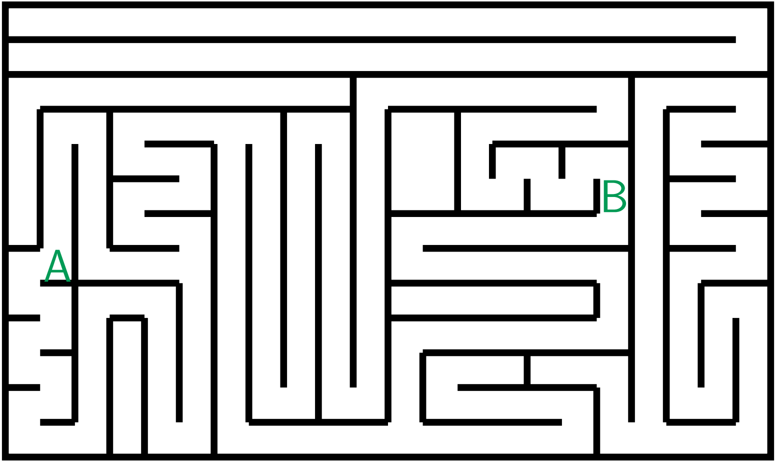

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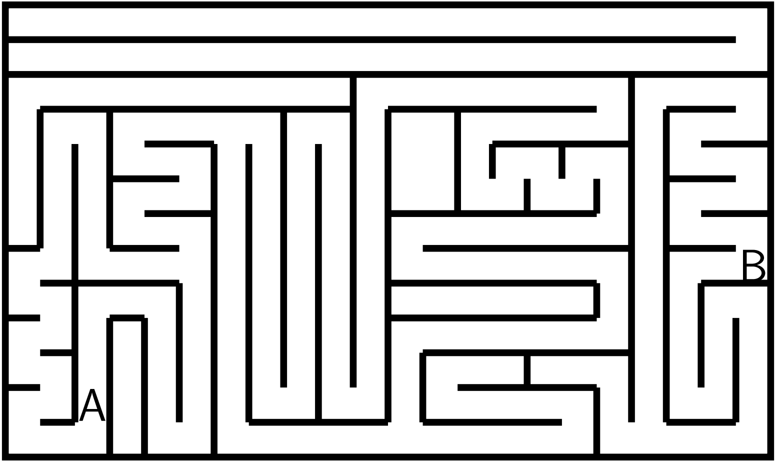

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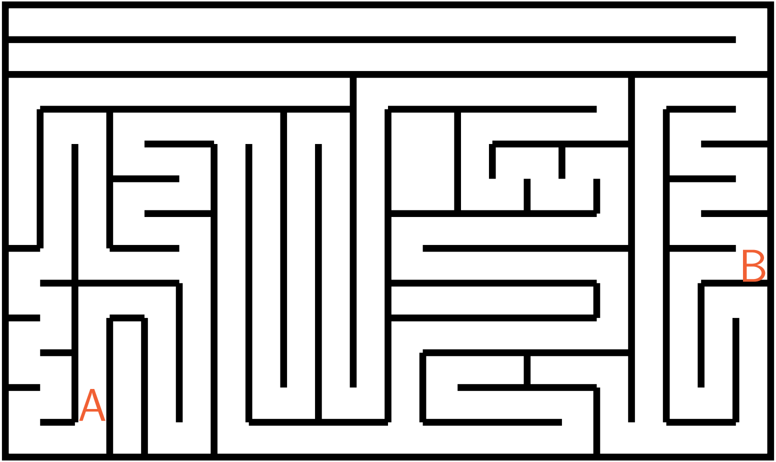

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  - otherwise, $\texttt{Find}(x) \neq \texttt{Find}(y)$
- $\texttt{Union}(x, y)$ merges two sets containing x and y

Preprocess(*maze*)

```
for each cell c in maze:
  MakeSet(c)
for each cell c in maze:
  for each neighbor n of c:
    Union(c, n)
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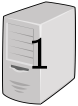
IsReachable(A, B)

```
return Find(A) = Find(B)
```

Building a Network

Building a Network

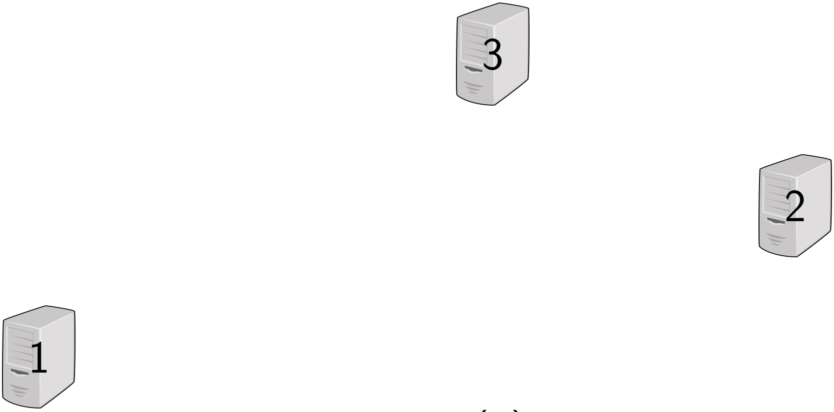


`MakeSet(1)`

Building a Network

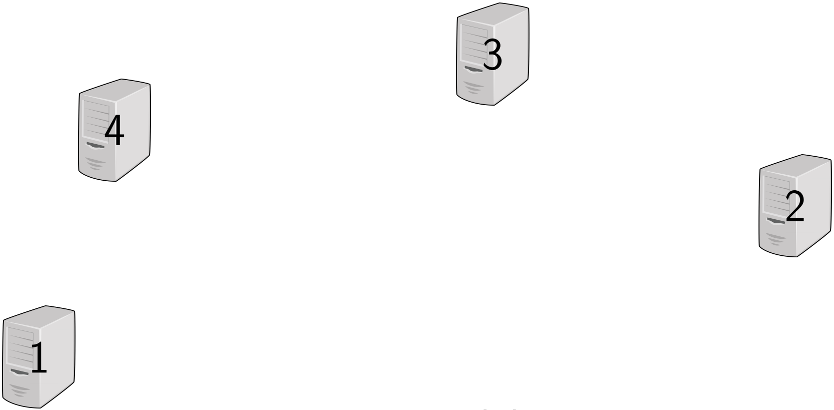
`MakeSet(2)`

Building a Network



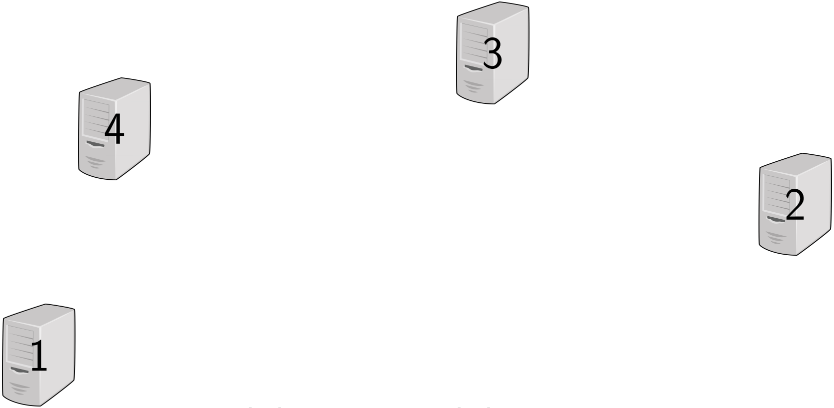
`MakeSet(3)`

Building a Network



`MakeSet(4)`

Building a Network



$$\texttt{Find}(1) = \texttt{Find}(2) \to \texttt{False}$$

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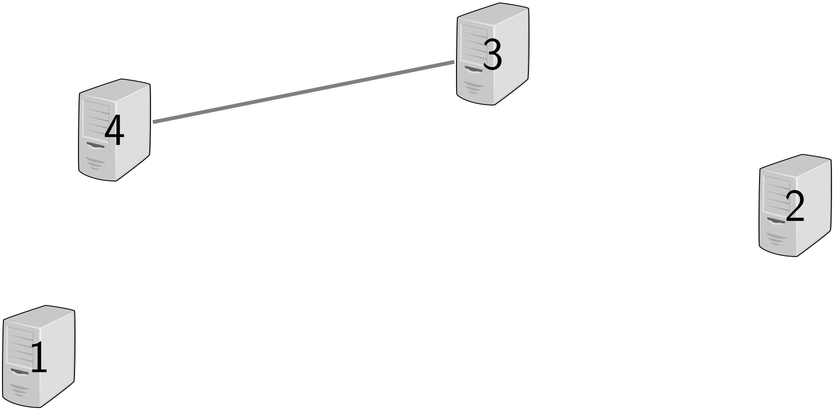

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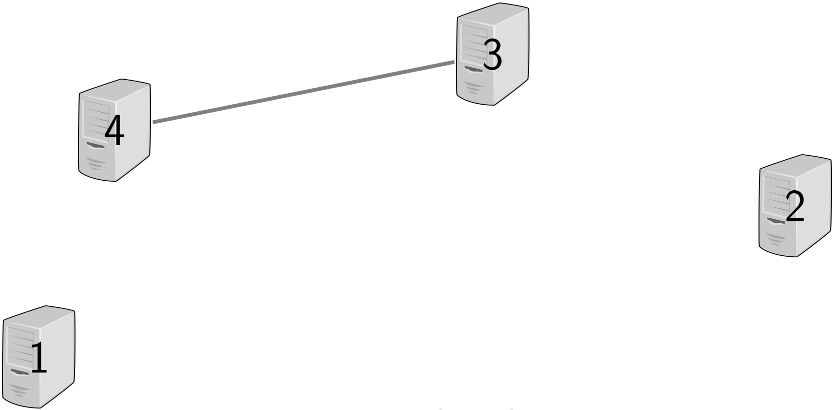
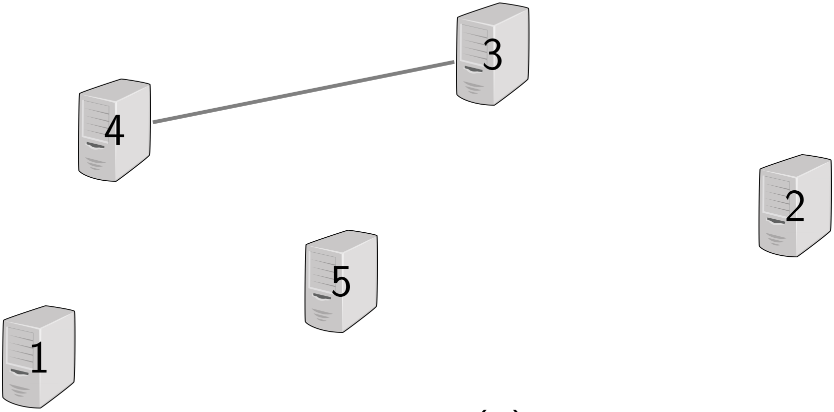


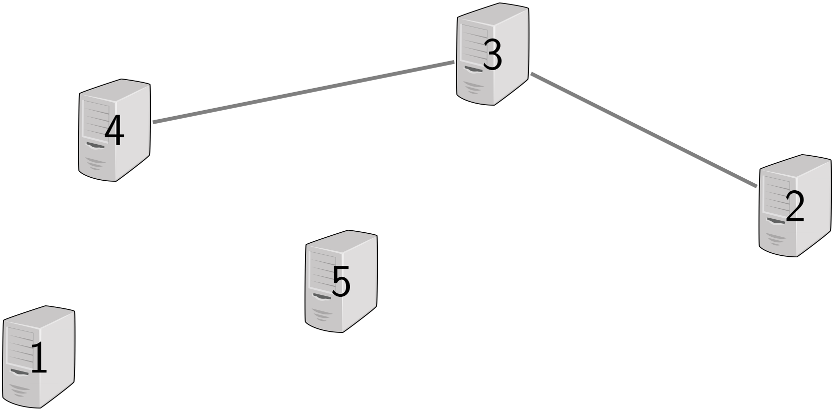
`Union(3, 4)`

Building a Network



`MakeSet(5)`

Building a Network



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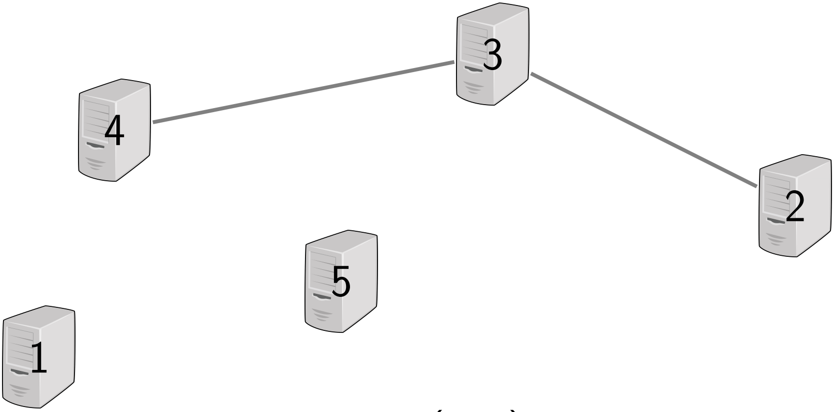
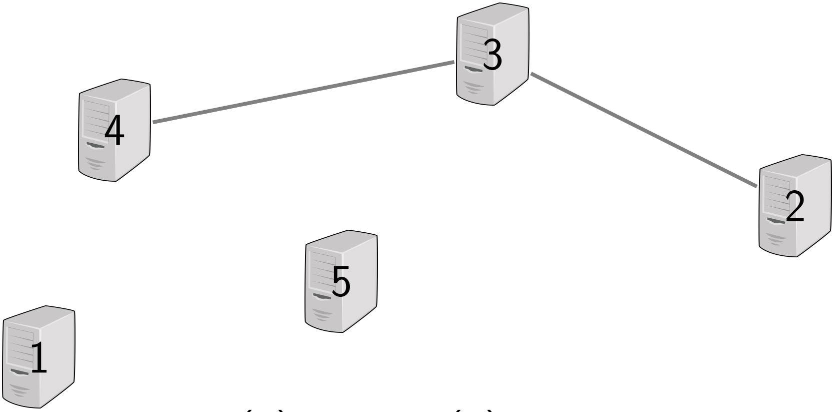


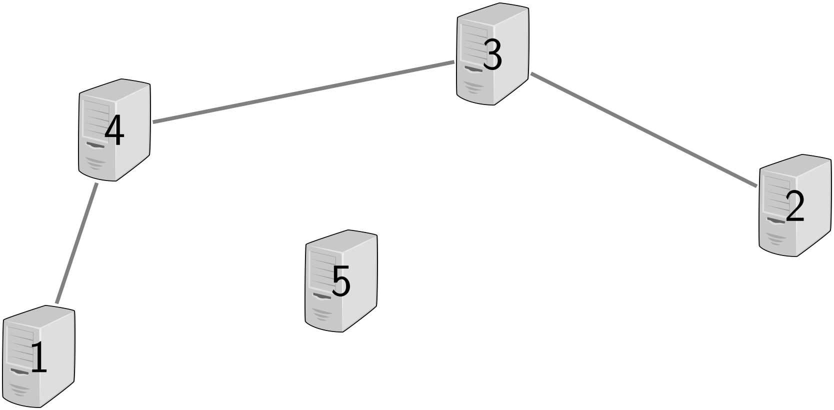
`Union(3, 2)`

Building a Network



$$\texttt{Find}(1) = \texttt{Find}(2) \to \texttt{False}$$

Building a Network



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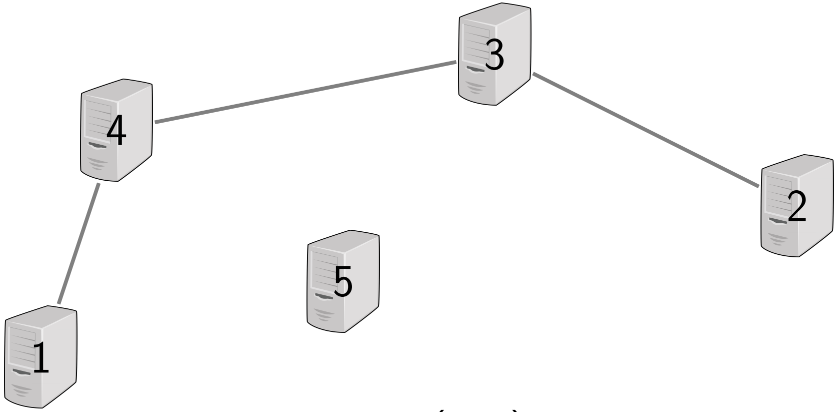
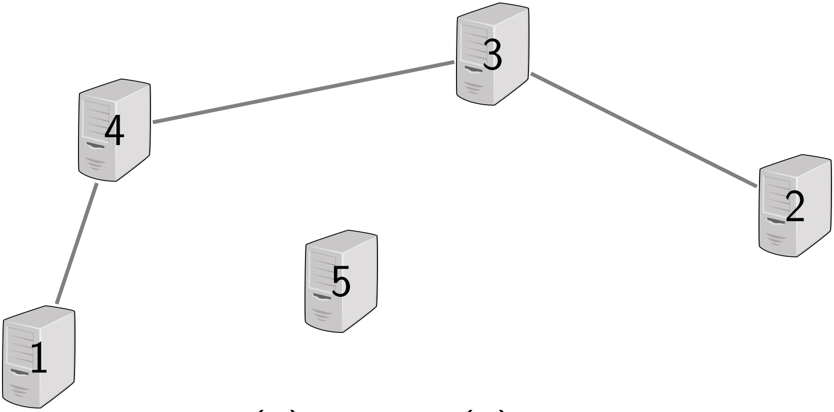


`Union(1, 4)`

Building a Network



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Outline

1. Overview
2. Naive Implementations

For simplicity, we assume that our n objects are just integers $1, 2, \ldots, n$.

Using the Smallest Element as ID

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- Use array `smallest`$[1 \dots n]$: `smallest`$[i]$ stores the smallest element in the set i belongs to

Example

$$\{9, 3, 2, 4, 7\} \quad \{5\} \quad \{6, 1, 8\}$$

	1	2	3	4	5	6	7	8	9
`smallest`	1	2	2	2	5	1	2	1	2

MakeSet(i)

```
smallest[i] ← i
```

Find(i)

```
return smallest[i]
```

MakeSet(i)

`smallest`$[i] \leftarrow i$

Find(i)

`return smallest`$[i]$

Running time: $O(1)$

Union(i, j)

```
i_id ← Find(i)
j_id ← Find(j)
if i_id = j_id:
  return
m ← min(i_id, j_id)
for k from 1 to n:
  if smallest[k] in {i_id, j_id}:
    smallest[k] ← m
```

`Union`(i, j)

$i\_id \leftarrow$ `Find`(i)
$j\_id \leftarrow$ `Find`(j)
`if` $i\_id = j\_id$`:`
  `return`
$m \leftarrow \min(i\_id, j\_id)$
`for` k `from 1 to` n`:`
  `if smallest`$[k]$ `in` $\{i\_id, j\_id\}$`:`
    `smallest`$[k] \leftarrow m$

Running time: $O(n)$

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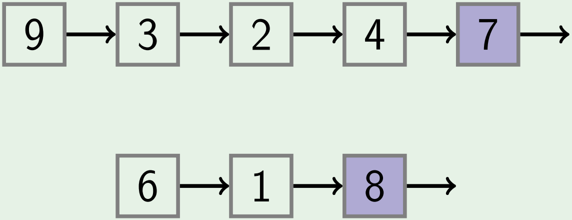
- What basic data structure allows for efficient merging?
- Linked list!
- Idea: represent a set as a linked list, use the list tail as ID of the set

Example: merging two lists



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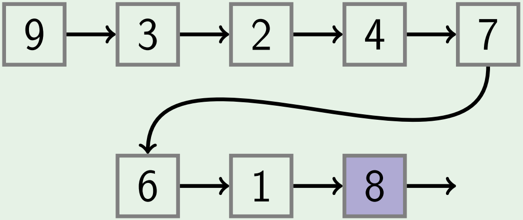

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- Cons:
  - Running time of `Find` is $O(n)$ as we need to traverse the list to find its tail
  - `Union`(x, y) works in time $O(1)$ only if we can get the tail of the list of x and the head of the list of y in constant time!

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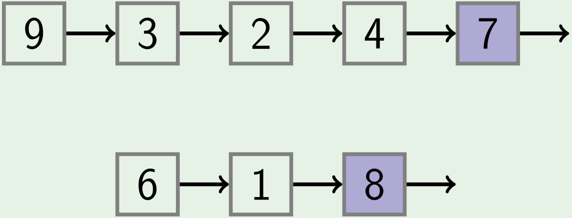

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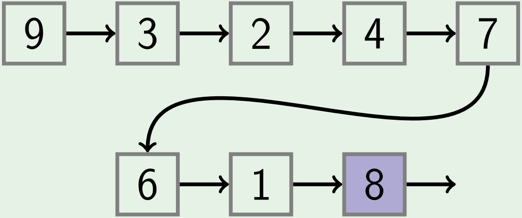

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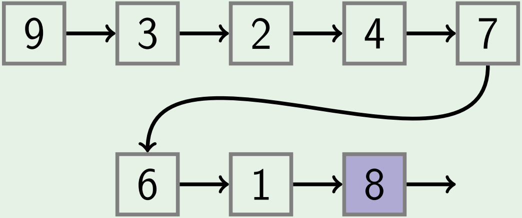

`Find`(9) goes through all elements

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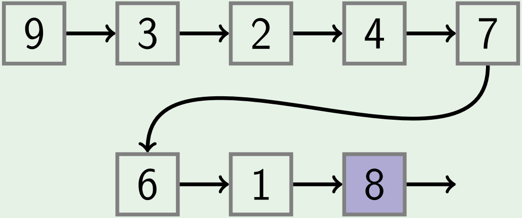

can we merge in a different way?

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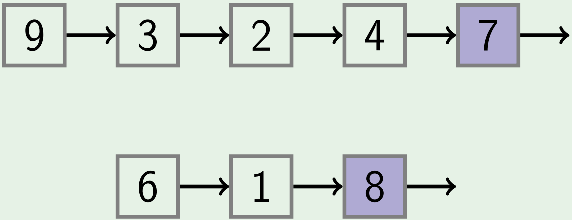

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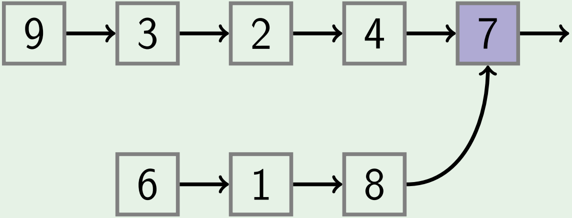

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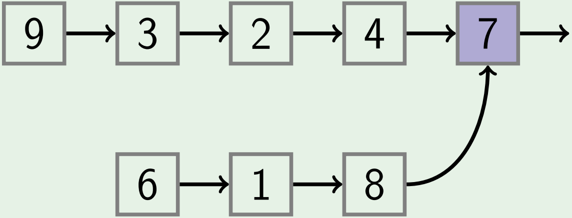
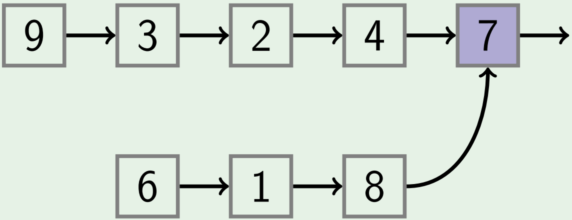


instead of a list we get a tree

Example: merging two lists



we'll see that representing sets as trees gives a very efficient implementation: nearly constant amortized time for all operations