Disjoint Sets: Efficient Implementations

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Data Structures Fundamentals
Algorithms and Data Structures

Outline

1. Trees
2. Union by Rank
3. Path Compression
4. Analysis

- Represent each set as a rooted tree

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- ID of a set is the root of the tree

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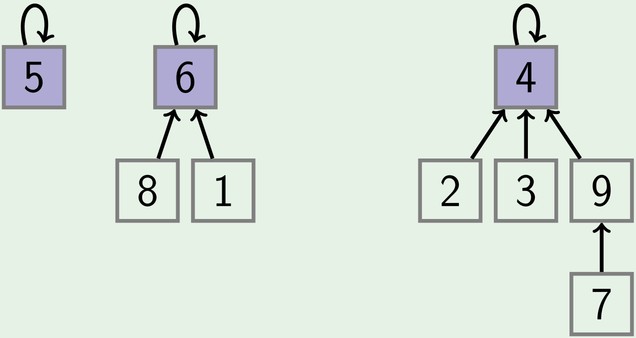
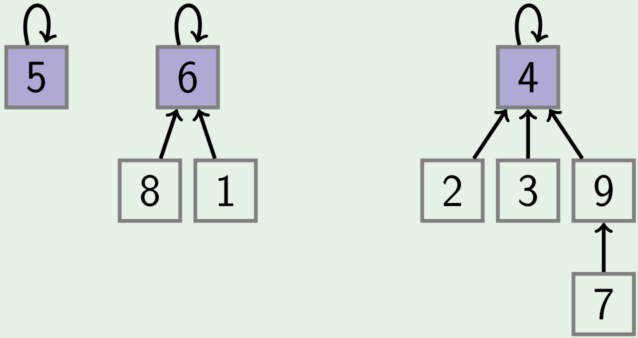
- ID of a set is the root of the tree
- Use array `parent`$[1 \ldots n]$: `parent`$[i]$ is the parent of i, or i if it is the root

Example



Example



	1	2	3	4	5	6	7	8	9
`parent`	6	4	4	4	5	6	9	6	4

`MakeSet(`i`)`

`parent[`i`]` $\leftarrow i$

MakeSet(i)

$\texttt{parent}[i] \leftarrow i$

Running time: $O(1)$

$\texttt{MakeSet}(i)$

$\texttt{parent}[i] \leftarrow i$

Running time: $O(1)$

$\texttt{Find}(i)$

$\texttt{while } i \neq \texttt{parent}[i]$:

$\quad i \leftarrow \texttt{parent}[i]$

$\texttt{return } i$

MakeSet(i)

```
parent[i] ← i
```

Running time: $O(1)$

Find(i)

```
while i ≠ parent[i]:
    i ← parent[i]
return i
```

Running time: $O(\text{tree height})$

- How to merge two trees?

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- Hang one of the trees under the root of the other one

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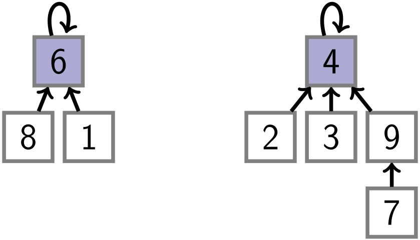
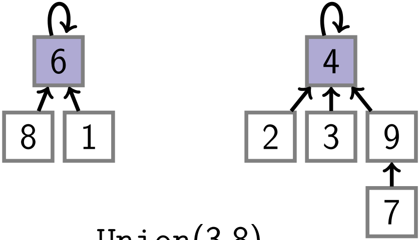
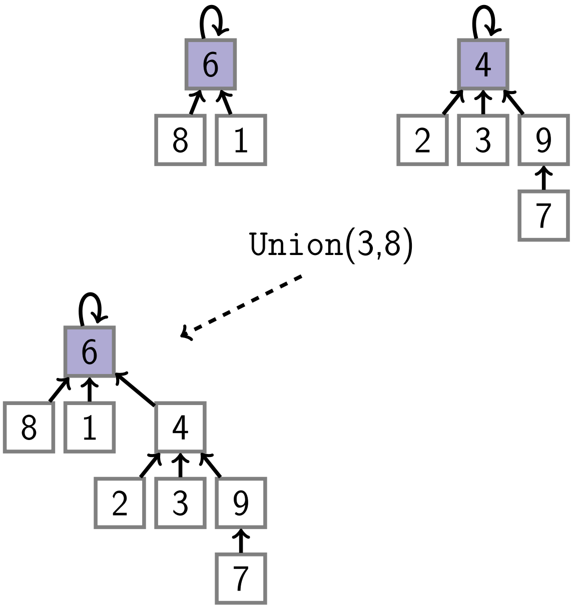
- Which one to hang?

- How to merge two trees?
- Hang one of the trees under the root of the other one
- Which one to hang?
- A shorter one, since we would like to keep the trees shallow


6
8
1
4
2
3
9
7



Union(3,8)


6
8
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Union(3,8)
6
8
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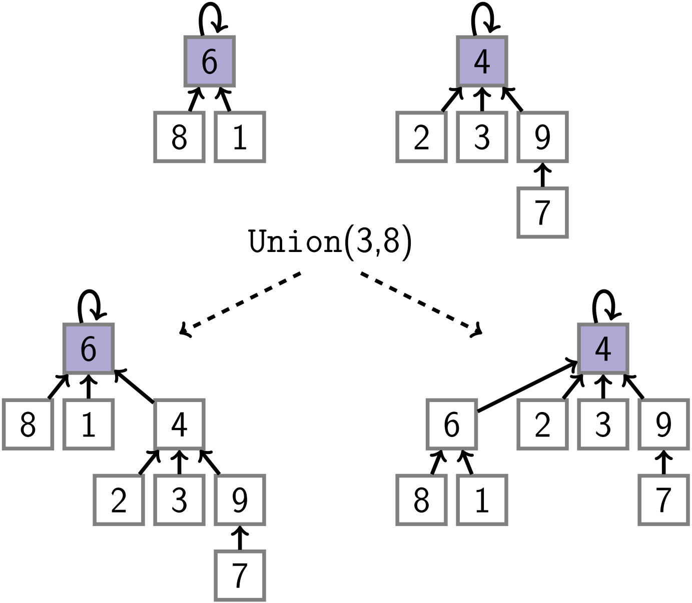
6
4
8
1
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9
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Union(3,8)
6
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1
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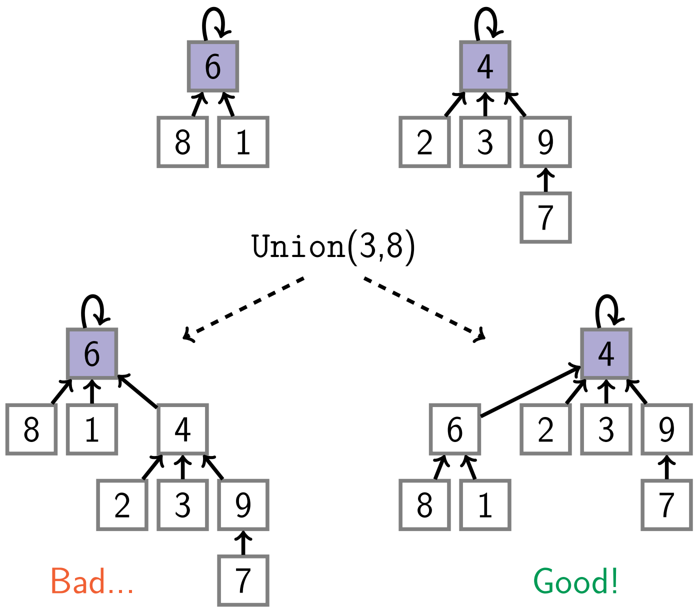
7
4
6
2
3
9
8
1
7


6
8
1
4
2
3
9
7
Union(3,8)
6
8
1
4
2
3
9
7
Bad...
4
6
2
3
9
8
1
7
Good!

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- (The reason we call it `rank`, but not `height` will become clear later)

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- To quickly find a height of a tree, we will keep the height of each subtree in an array `rank`$[1 \dots n]$: `rank`$[i]$ is the height of the subtree whose root is i
- (The reason we call it `rank`, but not `height` will become clear later)
- Hanging a shorter tree under a taller one is called a union by rank heuristic

MakeSet(i)

```
parent[i] ← i
rank[i] ← 0
```

Find(i)

```
while i ≠ parent[i]:
    i ← parent[i]
return i
```

Union(i, j)

```
i_id ← Find(i)
j_id ← Find(j)
if i_id = j_id:
  return
if rank[i_id] > rank[j_id]:
  parent[j_id] ← i_id
else:
  parent[i_id] ← j_id
  if rank[i_id] = rank[j_id]:
    rank[j_id] ← rank[j_id] + 1
```

Example

Query:

	1	2	3	4	5	6
`parent`						
`rank`						

Example

Query:

`MakeSet(1)`

`MakeSet(2)`

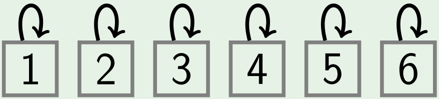
`...`

`MakeSet(6)`

	1	2	3	4	5	6
parent						
rank						

Example

Query:

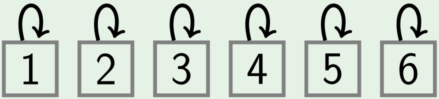


	1	2	3	4	5	6
parent	1	2	3	4	5	6
rank	0	0	0	0	0	0

Example

Query:

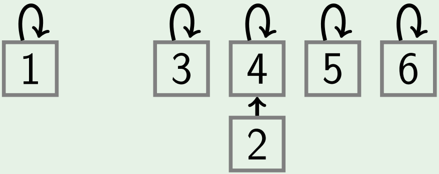
$\texttt{Union}(2, 4)$



	1	2	3	4	5	6
parent	1	2	3	4	5	6
rank	0	0	0	0	0	0

Example

Query:

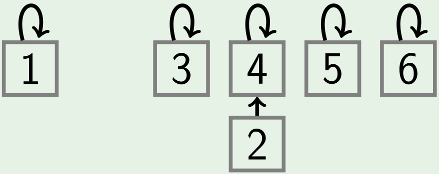


	1	2	3	4	5	6
`parent`	1	4	3	4	5	6
`rank`	0	0	0	1	0	0

Example

Query:

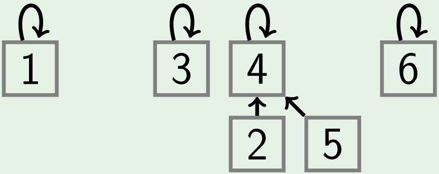
$\texttt{Union}(5, 2)$



	1	2	3	4	5	6
parent	1	4	3	4	5	6
rank	0	0	0	1	0	0

Example

Query:

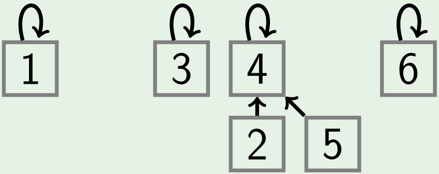


	1	2	3	4	5	6
`parent`	1	4	3	4	4	6
`rank`	0	0	0	1	0	0

Example

Query:

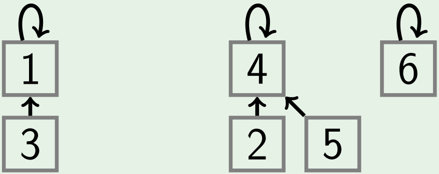
Union(3, 1)



	1	2	3	4	5	6
parent	1	4	3	4	4	6
rank	0	0	0	1	0	0

Example

Query:

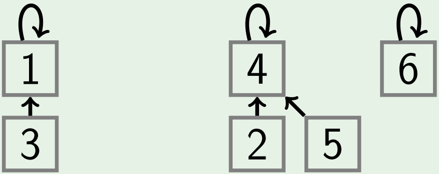


	1	2	3	4	5	6
`parent`	1	4	1	4	4	6
`rank`	1	0	0	1	0	0

Example

Query:

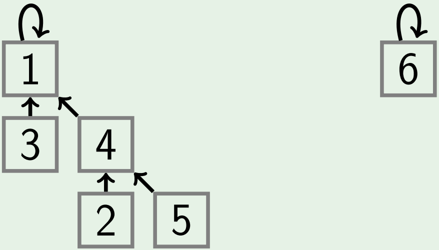
`Union`$(2, 3)$



	1	2	3	4	5	6
`parent`	1	4	1	4	4	6
`rank`	1	0	0	1	0	0

Example

Query:

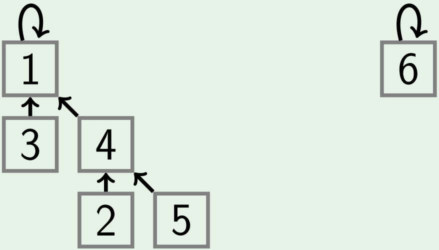


	1	2	3	4	5	6
`parent`	1	4	1	1	4	6
`rank`	2	0	0	1	0	0

Example

Query:

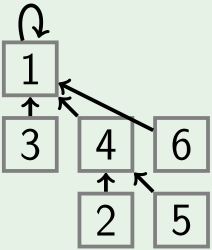
Union$(2, 6)$



	1	2	3	4	5	6
parent	1	4	1	1	4	6
rank	2	0	0	1	0	0

Example

Query:



	1	2	3	4	5	6
`parent`	1	4	1	1	4	1
`rank`	2	0	0	1	0	0

Important property: for any node i, rank[i] is equal to the height of the tree rooted at i

Lemma

The height of any tree in the forest is at most $\log_2 n$.

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Follows from the following lemma.

Lemma

Any tree of height k in the forest has at least 2^k nodes.

Proof

Induction on k.

- Base: initially, a tree has height 0 and one node: $2^0 = 1$.
- Step: a tree of height k results from merging two trees of height $k-1$. By induction hypothesis, each of two trees has at least 2^{k-1} nodes, hence the resulting tree contains at least 2^k nodes. $\square$

Summary

The union by rank heuristic guarantees that `Union` and `Find` work in time $O(\log n)$.

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Next part

We'll discover another heuristic that improves the running time to nearly constant!

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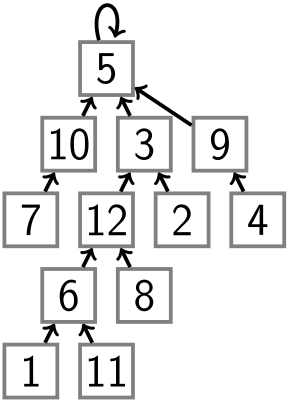
Path Compression: Intuition



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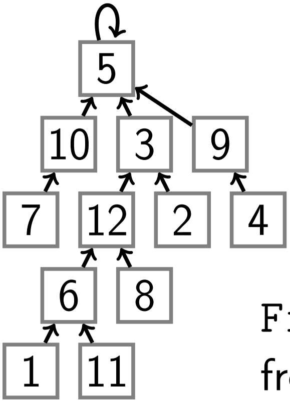
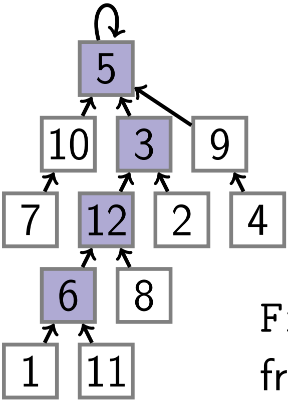


`Find`(6) traverses the path from 6 to the root

Path Compression: Intuition



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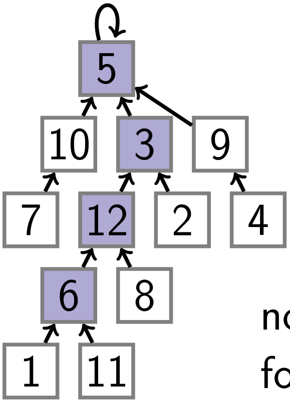
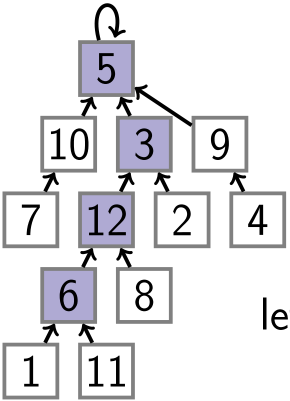


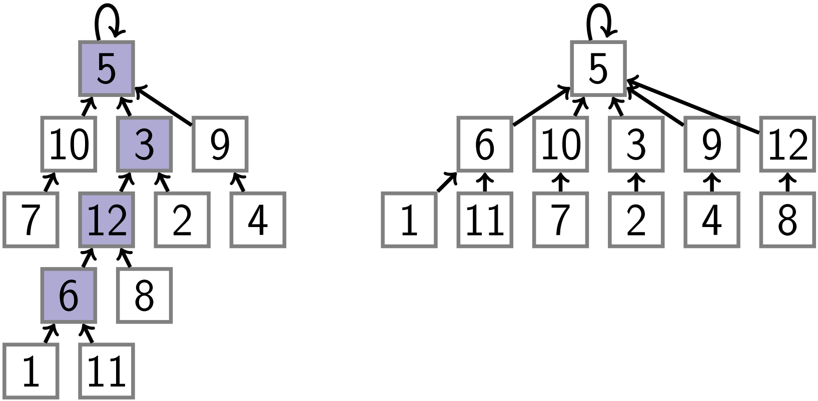
not only it finds the root for 6, it does so for all the nodes on this path

Path Compression: Intuition



let's not lose this useful info

Path Compression: Intuition



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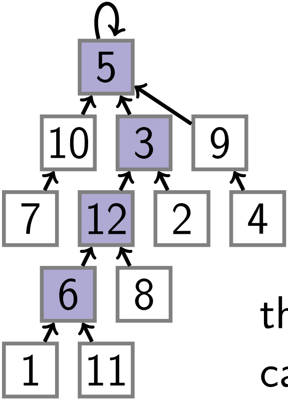

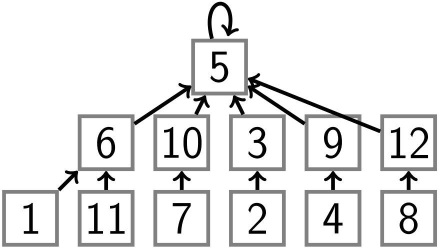

the resulting heuristic is called path compression

Find(i)

```
if i ≠ parent[i]:
    parent[i] ← Find(parent[i])
return parent[i]
```

Definition

The iterated logarithm of n, $\log^* n$, is the number of times the logarithm function needs to be applied to n before the result is less or equal than 1:

$$\log^* n = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + \log^*(\log n) & \text{if } n > 1 \end{cases}$$

Example

n	$\log^* n$
$n = 1$	0
$n = 2$	1
$n \in \{3, 4\}$	2
$n \in \{5, 6, \ldots, 16\}$	3
$n \in \{17, \ldots, 65536\}$	4
$n \in \{65537, \ldots, 2^{65536}\}$	5

Lemma

Assume that initially the data structure is empty. We make a sequence of m operations including n calls to `MakeSet`. Then the total running time is $O(m \log^* n)$.

In other words

The amortized time of a single operation is $O(\log^* n)$.

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Nearly constant!

For practical values of n, $\log^* n \leq 5$.

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Goal

Prove that when both union by rank heuristic and path compression heuristic are used, the average running time of each operation is nearly constant.

Height $\leq$ Rank

- When using path compression, `rank`$[i]$ is no longer equal to the height of the subtree rooted at i

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- Still, the height of the subtree rooted at i is at most `rank`[i]

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- When using path compression, `rank`[i] is no longer equal to the height of the subtree rooted at i
- Still, the height of the subtree rooted at i is at most `rank`[i]
- And it is still true that a root node of rank k has at least 2^k nodes in its subtree: a root node is not affected by path compression

Important Properties

1. There are at most $\frac{n}{2^k}$ nodes of rank k

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2. For any node i,
   $\texttt{rank}[i] < \texttt{rank}[\texttt{parent}[i]]$
3. Once an internal node, always an internal node

$T(\text{all calls to } \texttt{Find}) =$

$\#(i \to j) =$

$\#(i \to j\colon j \text{ is a root})+$

$\#(i \to j\colon\ \log^*(\texttt{rank}[i]) < \log^*(\texttt{rank}[j]))+$

$\#(i \to j\colon\ \log^*(\texttt{rank}[i]) = \log^*(\texttt{rank}[j]))$

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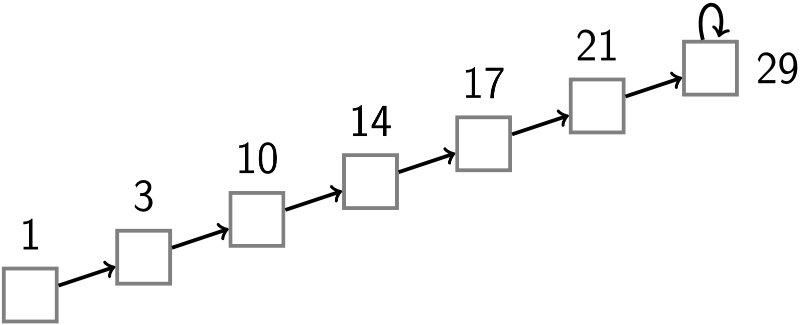

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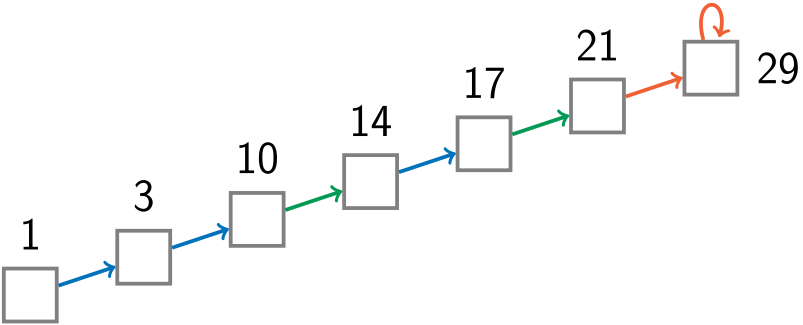

Claim

$\#(i \to j\colon j \text{ is a root}) \le O(m)$

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Proof

There are at most m calls to `Find`. $\square$

Claim

$$\#(i \to j\colon\ \log^*(\texttt{rank}[i]) < \log^*(\texttt{rank}[j]))$$
$$\leq O(m \log^* n)$$

Claim

$$\#(i \to j\colon\ \log^*(\texttt{rank}[i]) < \log^*(\texttt{rank}[j])) \le O(m \log^* n)$$

Proof

There are at most $\log^* n$ different values for $\log^*(\texttt{rank})$. $\square$

Claim

$$\#(i \to j\colon\ \log^*(\texttt{rank}[i]) = \log^*(\texttt{rank}[j])) \leq O(n \log^* n)$$

Proof

- assume $\texttt{rank}[i] \in \{k+1, \ldots, 2^k\}$

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$$\frac{n}{2^{k+1}} + \frac{n}{2^{k+2}} + \cdots \leq \frac{n}{2^k}$$

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- after a call to $\texttt{Find}(i)$, the node i is adopted by a new parent of strictly larger rank
- after at most 2^k calls to $\texttt{Find}(i)$, the parent of i will have rank from a different interval

Proof (Continued)

- there are at most $\frac{n}{2^k}$ nodes with rank in $\{k+1, \ldots, 2^k\}$

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- there are at most $\frac{n}{2^k}$ nodes with rank in $\{k+1, \ldots, 2^k\}$
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- the contribution of all the nodes with rank from this interval is at most $O(n)$

Proof (Continued)

- there are at most $\frac{n}{2^k}$ nodes with rank in $\{k+1, \ldots, 2^k\}$
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- the contribution of all the nodes with rank from this interval is at most $O(n)$
- the number of different intervals is $\log^* n$

Proof (Continued)

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- each of them contributes at most 2^k
- the contribution of all the nodes with rank from this interval is at most $O(n)$
- the number of different intervals is $\log^* n$
- thus, the contribution of all nodes is $O(n \log^* n)$ $\square$

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- Represent each set as a rooted tree
- Use the root of the set as its ID
- Union by rank heuristic: hang a shorter tree under the root of a taller one
- Path compression heuristic: when finding the root of a tree for a particular node, reattach each node from the traversed path to the root
- Amortized running time: $O(\log^* n)$ (constant for practical values of n)